

# CONSTRAINED VARIATIONAL CALCULUS: THE SECOND VARIATION (PART I)

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**ABSTRACT.** This paper is a direct continuation of [1]. The Hamiltonian aspects of the theory are further developed. Within the framework provided by the first paper, the problem of minimality for constrained calculus of variations is analyzed among the class of differentiable curves. A necessary and sufficient condition for minimality is proved.

*Keywords:* Constrained calculus of variations, minimality, second variation.

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## INTRODUCTION

This is the earlier of two papers concerned with a further analysis of the arguments of [1] on the topic of constrained calculus of variations.

Consider an abstract system  $\mathfrak{B}$ , subject to a set of differentiable and possibly non-holonomic constraints. In its essential features, the problem tackled in these papers is the one of finding, within the class of the admissible evolutions  $\gamma$  of  $\mathfrak{B}$  joining two given configurations, the one which *minimizes* the action functional

$$\mathcal{I}[\gamma] := \int_{\hat{\gamma}} \mathcal{L}(t, q^i(t), z^A(t)) dt$$

being  $\mathcal{L}(t, q^1, \dots, q^n, z^1, \dots, z^r)$  a function on the admissible velocity space of the system and  $\hat{\gamma}$  the lift of  $\gamma$  to the latter.

A preliminary step in this direction was just provided by [1], in which the stationarity conditions for the functional  $\mathcal{I}[\gamma]$  were studied through the analysis of its first variation. The algorithm resulted into a set of conditions characterizing stationary curves among the class of piecewise differentiable ones, i.e. among the class of continuous curves having a finite number of discontinuities in their first derivative. Furthermore, the analysis pointed out the equivalence between the given constrained Lagrangian problem and a *free* Hamiltonian one.

The present paper is instead focussed on establishing whether a given extremal curve provides a local minimum of the functional  $\mathcal{I}[\gamma]$ . For the time being, the argument is developed within the only class of differentiable curves, leaving to the latter paper all problems arising from the possible presence of corners.

As far as the present work is concerned, the demonstrative strategy may be briefly sketched as below.

On the first instance, in §2, taking a given extremal curve  $\gamma$  into account, a sufficient condition for minimality is proved by means of the analysis of the second variation of the action functional.

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In this connection, the geometrical tools developed in [1] prove very useful in order to simplify the matter. The argument turns out to be subject to the solvability along the whole of  $\gamma$  of a Riccati-like differential equation. Unfortunately, because of the non-linearity of the latter, Cauchy theorem can only guarantee the existence of purely local solutions. Hence, at this stage, the global solvability is leaved among the hypotheses.

Subsequently, in §3, the attention is shifted onto Jacobi vector fields. They represent a special class of infinitesimal deformations which link families of extremal curves. They are used to investigate the processes of focalization and, by means of the further concept of *conjugate point*, to give a necessary condition for minimality.

At last, in §4, the sufficient condition and the necessary one are merged together, showing that the lack of conjugate points along the extremal curve  $\gamma$  implies the solvability of the above non-linear differential equation along the whole of it.

## 1. GEOMETRIC SETUP

**1.1. Preliminaries.** This Section is meant to be a reference tool consisting of a brief review of those contents of [1] that will be involved in the subsequent discussion. All results are stated without any proofs nor comments. Of course, the reader is referred to [1] and references therein for a thorough description of the subject.

(i) Let  $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$  denote a fibre bundle over the real line, henceforth called the *event space*, and referred to local fibered coordinates  $t, q^1, \dots, q^n$ .

Every section  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$  will be interpreted as the evolution of an abstract system  $\mathfrak{B}$ , parameterized in terms of the independent variable  $t$ . The first jet-bundle  $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ , referred to local jet-coordinates  $t, q^i, \dot{q}^i$ , will be called the *velocity space*. The first jet-extension of  $\gamma$  will be denoted by  $j_1(\gamma) : \mathbb{R} \rightarrow j_1(\mathcal{V}_{n+1})$ .

We shall indicate by  $V(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$  the vertical bundle over  $\mathcal{V}_{n+1} \rightarrow \mathbb{R}$  and by  $V^*(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$  the associated *dual* bundle, commonly referred to as the *phase space*. By definition, the latter is canonically isomorphic to the quotient of the cotangent bundle  $T^*(\mathcal{V}_{n+1})$  by the equivalence relation

$$\sigma \sim \sigma' \iff \begin{cases} \pi(\sigma) = \pi(\sigma') \\ \sigma - \sigma' \propto dt|_{\pi(\sigma)} \end{cases}$$

To make it easier, we preserve the notation  $\langle \cdot, \cdot \rangle$  for the pairing between  $V(\mathcal{V}_{n+1})$  and  $V^*(\mathcal{V}_{n+1})$ . In this way, every local coordinate system  $t, q^i$  in  $\mathcal{V}_{n+1}$  induces fibered coordinates  $t, q^i, p_i$  in  $V^*(\mathcal{V}_{n+1})$ , uniquely defined by the requirement

$$p_i(\hat{\sigma}) := \left\langle \hat{\sigma}, \left( \frac{\partial}{\partial q^i} \right)_{\pi(\hat{\sigma})} \right\rangle \quad \forall \hat{\sigma} \in V^*(\mathcal{V}_{n+1})$$

(ii) The pull-back of the phase space through the map  $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ , described by the commutative diagram

$$(1.1) \quad \begin{array}{ccc} \mathcal{C}(j_1(\mathcal{V}_{n+1})) & \xrightarrow{\kappa} & V^*(\mathcal{V}_{n+1}) \\ \zeta \downarrow & & \downarrow \pi \\ j_1(\mathcal{V}_{n+1}) & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array}$$

gives rise to a manifold  $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$ , henceforth referred to as the *contact bundle*.

We shall refer it to fibered coordinates  $t, q^i, \dot{q}^i, p_i$ , related in an obvious way to those in  $j_1(\mathcal{V}_{n+1})$  and in  $V^*(\mathcal{V}_{n+1})$ . Every  $\sigma \in \mathcal{C}(j_1(\mathcal{V}_{n+1}))$  will be called a *contact 1-form* over  $j_1(\mathcal{V}_{n+1})$ . Notice that, by construction,  $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$  is at the same time a vector bundle over  $j_1(\mathcal{V}_{n+1})$ , with projection  $\zeta$ , and an affine bundle over  $V^*(\mathcal{V}_{n+1})$ , with projection  $\kappa$ .

The bundle  $\mathcal{C}(j_1(\mathcal{V}_{n+1})) \xrightarrow{\zeta} j_1(\mathcal{V}_{n+1})$  is canonically isomorphic to the vector subbundle of the cotangent space  $T^*(j_1(\mathcal{V}_{n+1}))$  locally spanned by the 1-forms

$$(1.2) \quad \omega^i := dq^i - \dot{q}^i dt \quad i = 1, \dots, n$$

With this identification, for every  $\sigma \in \mathcal{C}(j_1(\mathcal{V}_{n+1}))$  the coordinates  $p_i(\sigma)$  coincide with the components involved in the representation

$$(1.3) \quad \sigma = p_i(\sigma) \omega^i|_{\zeta(\sigma)}$$

Eventually, it is worth recalling that the contact bundle carries a distinguished *Liouville 1-form*  $\Theta$ , locally expressed as <sup>1</sup>

$$(1.4) \quad \Theta = p_i \omega^i = p_i(dq^i - \dot{q}^i dt)$$

(iii) The presence of differentiable constraints restricting the admissible evolutions of the system is formally accounted for by a commutative diagram of the form

$$(1.5a) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{V}_{n+1} & \xlongequal{\quad} & \mathcal{V}_{n+1} \end{array}$$

where

- $\mathcal{A} \xrightarrow{\pi} \mathcal{V}_{n+1}$  is a fiber bundle, representing the totality of *admissible velocities*;
- the map  $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$  is an imbedding;
- a section  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$  is admissible if and only if its jet-extension  $j_1(\gamma)$  factors through  $\mathcal{A}$ , i.e. if and only if there exists a section  $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$  satisfying  $j_1(\gamma) = i \cdot \hat{\gamma}$ . Under the stated circumstance the section  $\hat{\gamma}$ , commonly referred to as the *lift* of  $\gamma$ , is called an admissible section of  $\mathcal{A}$ .

Referring the submanifold  $\mathcal{A}$  to fibered local coordinates  $t, q^1, \dots, q^n, z^1, \dots, z^r$ , the imbedding  $i : \mathcal{A} \rightarrow j_1(\mathcal{V}_{n+1})$  is thus locally represented as

$$(1.5b) \quad \dot{q}^i = \psi^i(t, q^1, \dots, q^n, z^1, \dots, z^r) \quad i = 1, \dots, n$$

while the admissibility conditions for a section  $\hat{\gamma} : q^i = q^i(t), z^A = z^A(t)$  read

$$(1.6) \quad \frac{dq^i}{dt} = \psi^i(t, q^1(t), \dots, q^n(t), z^1(t), \dots, z^r(t))$$

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<sup>1</sup>For simplicity, the same notation is used for the 1-forms (1.2) as well as for their pull-back on  $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$ .

Every section  $\sigma : \mathcal{V}_{n+1} \rightarrow \mathcal{A}$  is called a *control*<sup>2</sup> for the system. The composite map  $i \cdot \sigma$  is called an *admissible velocity field*. In local coordinates we have the representations

$$(1.7a) \quad \sigma : \quad z^A = z^A(t, q^1, \dots, q^n)$$

$$(1.7b) \quad i \cdot \sigma : \quad \dot{q}^i = \psi^i(t, q^i, z^A(t, q^i))$$

A section  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$  and a control  $\sigma : \mathcal{V}_{n+1} \rightarrow \mathcal{A}$  are said to *belong* to each other if and only if the composition  $\sigma \cdot \gamma$  coincides with the lift  $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$ .

(iv) The concepts of vertical vector and contact 1-form are easily adapted to the submanifold  $\mathcal{A}$ : as usual, the vertical bundle  $V(\mathcal{A})$  is the kernel of the push-forward  $T(\mathcal{A}) \xrightarrow{\pi_*} T(\mathcal{V}_{n+1})$  while the contact bundle  $\mathcal{C}(\mathcal{A})$  is the pull-back on  $\mathcal{A}$  of the contact bundle  $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$ , as expressed by the commutative diagram

$$(1.8) \quad \begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \xrightarrow{\hat{i}} & \mathcal{C}(j_1(\mathcal{V}_{n+1})) \\ \zeta \downarrow & & \downarrow \zeta \\ \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \end{array}$$

The manifolds  $V(\mathcal{A})$ ,  $\mathcal{C}(\mathcal{A})$  will be respectively referred to coordinates  $t, q^i, z^A, v^A$  and  $t, q^i, z^A, p_i$ . In this way, setting

$$(1.9) \quad \tilde{\omega}^i := i^*(\omega^i) = dq^i - \psi^i(t, q^k, z^A) dt$$

we have the representations

$$X = v^A(X) \left( \frac{\partial}{\partial z^A} \right)_{\zeta(X)} \quad \forall X \in V(\mathcal{A}); \quad \sigma = p_i(\sigma) \tilde{\omega}^i|_{\zeta(\sigma)} \quad \forall \sigma \in \mathcal{C}(\mathcal{A})$$

The pull-back  $\tilde{\Theta} := \hat{i}^*(\Theta)$  of the 1-form (1.4) will be called the *Liouville 1-form* of  $\mathcal{C}(\mathcal{A})$ . In coordinates, eqs. (1.4), (1.9) provide the representation

$$(1.10) \quad \tilde{\Theta} = p_i \tilde{\omega}^i = p_i (dq^i - \psi^i dt)$$

Finally, setting  $\hat{\kappa} := \kappa \cdot \hat{i}$ , a straightforward composition of diagrams (1.8), (1.1) yields the bundle morphism

$$(1.11) \quad \begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \xrightarrow{\hat{\kappa}} & V^*(\mathcal{V}_{n+1}) \\ \zeta \downarrow & & \downarrow \pi \\ \mathcal{A} & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array}$$

expressing the contact bundle  $\mathcal{C}(\mathcal{A})$  as a fibre bundle over  $V^*(\mathcal{V}_{n+1})$ , identical to the pull-back of the latter through the map  $\mathcal{A} \xrightarrow{\pi} \mathcal{V}_{n+1}$ .

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<sup>2</sup>The term *control* is intuitively clear since every assignment of the section  $\sigma$  determines the evolution of the system from a given initial configuration through the solution of a corresponding Cauchy problem for the ordinary differential equations (1.6).

**1.2. Vector bundles along sections.** From here on we shall stick on the geometrical environment based on diagram (1.8).

(i) Given any admissible section  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$  lifting to a section  $\hat{\gamma} : \mathbb{R} \rightarrow \mathcal{A}$ , we denote by  $V(\gamma) \xrightarrow{t} \mathbb{R}$  the vector bundle formed by the totality of vertical vectors along  $\gamma$  and by  $A(\hat{\gamma}) \xrightarrow{t} \mathbb{R}$  the analogous bundle formed by the totality of vectors along  $\hat{\gamma}$  annihilating the 1-form  $dt$ .

Both bundles will be referred to fibered coordinates — respectively  $t, u^i$  in  $V(\gamma)$  and  $t, u^i, v^A$  in  $A(\hat{\gamma})$  — according to the prescriptions

$$(1.12a) \quad X \in V(\gamma) \iff X = u^i(X) \left( \frac{\partial}{\partial q^i} \right)_{\gamma(t(X))}$$

$$(1.12b) \quad \hat{X} \in A(\hat{\gamma}) \iff \hat{X} = u^i(\hat{X}) \left( \frac{\partial}{\partial q^i} \right)_{\hat{\gamma}(t(\hat{X}))} + v^A(\hat{X}) \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}(t(\hat{X}))}$$

Recalling that the first jet-extension of the section  $\gamma$  is denoted by  $j_1(\gamma)$ , we have the following

**Proposition 1.1.** *The first jet-bundle  $j_1(V(\gamma))$  is canonically isomorphic to the space of vectors along  $j_1(\gamma)$  annihilating the 1-form  $dt$ . In jet-coordinates, the identification is expressed by the relation*

$$Z \in j_1(V(\gamma)) \iff Z = u^i(Z) \left( \frac{\partial}{\partial q^i} \right)_{j_1(\gamma)(t(Z))} + \dot{u}^i(Z) \left( \frac{\partial}{\partial \dot{q}^i} \right)_{j_1(\gamma)(t(Z))}$$

The push-forward of the imbedding  $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$ , restricted to the subspace  $A(\hat{\gamma}) \subset T(\mathcal{A})$ , makes the latter into a subbundle of  $j_1(V(\gamma))$ . This gives rise to a fibered morphism

$$(1.13a) \quad \begin{array}{ccc} A(\hat{\gamma}) & \xrightarrow{i_*} & j_1(V(\gamma)) \\ \pi_* \downarrow & & \downarrow \pi_* \\ V(\gamma) & \xlongequal{\quad} & V(\gamma) \end{array}$$

expressed in coordinates as

$$(1.13b) \quad \dot{u}^i = \left( \frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} u^k + \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} v^A$$

The kernel of the projection  $A(\hat{\gamma}) \xrightarrow{\pi_*} V(\gamma)$ , denoted by  $V(\hat{\gamma})$ , will be called the *vertical bundle* along  $\hat{\gamma}$ . In fiber coordinates,  $V(\hat{\gamma})$  has local equation  $u^i = 0$ , i.e. it coincides with the subbundle of  $A(\hat{\gamma})$  formed by the totality of vectors of the form  $\hat{X} = v^A(\hat{X}) \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}(t(\hat{X}))}$ .

The main interest of Proposition 1.1 stems from the study of the *infinitesimal deformations*. Still referring to [1] for details, we recall that an admissible infinitesimal deformation of an admissible section  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$  is expressed as a section  $X : \mathbb{R} \rightarrow V(\gamma)$  — i.e. as a vertical vector field along  $\gamma$  — whose jet-extension  $j_1(X)$  “factors” through diagram (1.13a), namely such that there exists a section  $\hat{X} : \mathbb{R} \rightarrow A(\hat{\gamma})$  satisfying  $j_1(X) = i_* \hat{X}$ .

Under the stated circumstance the section  $\hat{X}$ , viewed as a vector field along  $\hat{\gamma}$ , is called an admissible infinitesimal deformation of  $\hat{\gamma}$ . Conversely, a necessary and sufficient condition for a section  $\hat{X} : \mathbb{R} \rightarrow A(\hat{\gamma})$  to represent an admissible infinitesimal deformation of  $\hat{\gamma}$  is the validity of the relation

$$(1.14a) \quad i_* \hat{X} = j_1(\pi_*(\hat{X}))$$

In coordinates, setting  $\hat{X} = X^i(t) \left( \frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + \Gamma^A(t) \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$  and recalling the representation (1.13b), eq. (1.14a) takes the familiar form

$$(1.14b) \quad \frac{dX^i}{dt} = \left( \frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} X^k + \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \Gamma^A$$

commonly referred to as the *variational equation*.

The previous arguments point out the perfectly symmetric roles played by diagram (1.5a) in the study of the admissible *evolutions* and by diagram (1.13a) in the study of the infinitesimal *deformations*, thus enforcing the idea that the latter context is essentially a “linearized counterpart” of the former one.

(ii) Let  $V^*(\gamma) \xrightarrow{t} \mathbb{R}$  be the *dual* of the vertical bundle  $V(\gamma) \xrightarrow{t} \mathbb{R}$ , naturally isomorphic to the pull-back on  $\gamma$  of the phase space  $V^*(\mathcal{V}_{n+1})$  described by the commutative diagram

$$(1.15) \quad \begin{array}{ccc} V^*(\gamma) & \longrightarrow & V^*(\mathcal{V}_{n+1}) \\ t \downarrow & & \downarrow \pi \\ \mathbb{R} & \xrightarrow{\gamma} & \mathcal{V}_{n+1} \end{array}$$

We preserve the notation  $\langle \cdot, \cdot \rangle$  for the pairing between  $V(\gamma)$  and  $V^*(\gamma)$ . The elements of  $V^*(\gamma)$  will be called the *virtual 1-forms* along  $\gamma$ . More generally, each element belonging to a fibered tensor product  $V(\gamma) \otimes_{\mathbb{R}} V^*(\gamma) \otimes_{\mathbb{R}} \cdots$  of copies of  $V(\gamma)$  and  $V^*(\gamma)$  in any order will be called a *virtual tensor* along  $\gamma$ .

Notice that, according to the stated definition, a virtual 1-form  $\hat{\lambda}$  at a point  $\gamma(t)$  is not a 1-form in the ordinary sense, but an *equivalence class* of 1-forms under the relation

$$(1.16) \quad \lambda \sim \lambda' \iff \lambda - \lambda' \propto (dt)_{\gamma(t)}$$

The equivalence class determined by the 1-form  $dq^i$  will be denoted by  $\hat{\omega}^i$ . With this notation, every section  $W : \mathbb{R} \rightarrow V(\gamma) \otimes_{\mathbb{R}} V^*(\gamma) \otimes_{\mathbb{R}} \cdots$  is locally expressed as

$$(1.17) \quad W = W^i_{j\dots}(t) \left( \frac{\partial}{\partial q^i} \right)_{\gamma} \otimes \hat{\omega}^j \otimes \cdots$$

(iii) Given an admissible evolution  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ , an *infinitesimal control* along  $\gamma$  is a linear section  $h : V(\gamma) \rightarrow A(\hat{\gamma})$ . The image  $\mathcal{H}(\hat{\gamma}) := h(V(\gamma))$  is called the *horizontal distribution* along  $\hat{\gamma}$  induced by  $h$ . Every section  $\hat{X} : \mathbb{R} \rightarrow A(\hat{\gamma})$  satisfying  $\hat{X}(t) \in \mathcal{H}(\hat{\gamma}) \ \forall t \in \mathbb{R}$  is called a horizontal section of  $A(\hat{\gamma})$ .

In fiber coordinates an infinitesimal control is locally represented as

$$(1.18) \quad v^A = h_i^A(t) u^i$$

The associated horizontal distribution is locally spanned by the vector fields

$$(1.19) \quad \tilde{\partial}_i := h \left[ \left( \frac{\partial}{\partial q^i} \right)_{\gamma} \right] = \left( \frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + h_i^A \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$$

Every section  $X = X^i(t) \left( \frac{\partial}{\partial q^i} \right)_{\gamma}$  of  $V(\gamma)$  may be lifted to a horizontal section  $h(X) = X^i \tilde{\partial}_i$  of  $A(\hat{\gamma})$ . More generally, every section  $\hat{X} = X^i(t) \left( \frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + \Gamma^A(t) \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$  of  $A(\hat{\gamma})$  may be uniquely decomposed into the sum of a horizontal and a vertical part, respectively described by the equations

$$(1.20a) \quad \mathcal{P}_H(\hat{X}) := h(\pi_*(\hat{X})) = X^i \tilde{\partial}_i$$

$$(1.20b) \quad \mathcal{P}_V(\hat{X}) = \hat{X} - \mathcal{P}_H(\hat{X}) = (\Gamma^A - X^i h_i^A) \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$$

(iv) A section  $X : \mathbb{R} \rightarrow V(\gamma)$  is said to be  $h$ -transported along  $\gamma$  if the horizontal lift  $h(X)$  is an admissible infinitesimal deformation of  $\hat{\gamma}$ , i.e. if it satisfies the condition  $i_* \cdot h(X) = j_1(X)$ .

In coordinates, this amounts to the requirement

$$(1.21) \quad \frac{dX^i}{dt} = \left[ \left( \frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} + h_k^A \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \right] X^k = X^k \tilde{\partial}_k \psi^i$$

By Cauchy theorem, the  $h$ -transported sections form an  $n$ -dimensional vector space  $V_h$ , isomorphic to each fiber  $V(\gamma)|_t$ . This provides a *trivialization* of the vector bundle  $V(\gamma) \rightarrow \mathbb{R}$ , summarized into the identification  $V(\gamma) \simeq \mathbb{R} \times V_h$ .

The dual space  $V_h^*$  gives rise to an analogous trivialization  $V^*(\gamma) \simeq \mathbb{R} \times V_h^*$ .

(v) The notion of  $h$ -transport induces a  $\mathbb{R}$ -linear *absolute time derivative* for vertical vector fields along  $\gamma$ . The operation may be extended to a derivation of the algebra of virtual tensor fields along  $\gamma$ , commuting with contractions.

In coordinates, introducing the *temporal connection coefficients*

$$(1.22) \quad \tau_k^i := -\tilde{\partial}_k(\psi^i) = -\left( \frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} - h_k^A \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}}$$

we have the representation

$$(1.23a) \quad \frac{D}{Dt} \left[ W^i_{j\dots}(t) \left( \frac{\partial}{\partial q^i} \right)_{\gamma} \otimes \hat{\omega}^j \otimes \dots \right] := \frac{DW^i_{j\dots}}{Dt} \left( \frac{\partial}{\partial q^i} \right)_{\gamma} \otimes \hat{\omega}^j \otimes \dots$$

with

$$(1.23b) \quad \frac{DW^i_{j\dots}}{Dt} = \frac{dW^i_{j\dots}}{dt} + \tau_k^i W^k_{j\dots} - \tau_j^k W^i_{k\dots} + \dots$$

The algorithm is greatly simplified referring the fibers of both bundles  $V(\gamma)$ ,  $V^*(\gamma)$  to  $h$ -transported dual bases  $e_{(a)} = e_{(a)}^i \left( \frac{\partial}{\partial q^i} \right)_{\gamma}$ ,  $e^{(a)} = e^{(a)}_i \hat{\omega}^i$ .

Setting  $W = W^a_{b\dots} e_{(a)} \otimes e^{(b)} \otimes \dots$  (with  $W^a_{b\dots} = W^i_{j\dots} e^{(a)}_i e^{(b)}_j \dots$ ) we have the representation

$$(1.24) \quad \frac{DW}{Dt} = \frac{dW^a_{b\dots}}{dt} e_{(a)} \otimes e^{(b)} \otimes \dots$$

(vi) In view of eqs. (1.20a,b), every infinitesimal deformation  $\hat{X}$  of the section  $\hat{\gamma}$  admits a unique representation of the form  $\hat{X} = h(X) + Y$ , where:

- $X = \pi_* \hat{X} := X^i \left( \frac{\partial}{\partial q^i} \right)_{\gamma}$  is a vertical field along  $\gamma$ , namely the unique infinitesimal deformation of  $\gamma$  lifting to  $\hat{X}$ ;
- $Y = \mathcal{P}_V(\hat{X}) := Y^A \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$  is a vertical vector field along  $\hat{\gamma}$ .

On account of eqs. (1.23a,b), the variational equation (1.14b) takes then the form

$$\frac{dX^i}{dt} = X^k \left( \frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} + \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} (X^k h_k^A + Y^A)$$

written more conveniently as

$$(1.25a) \quad \frac{DX^i}{Dt} = \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} Y^A$$

or also, in self-transported bases

$$(1.25b) \quad \frac{dX^a}{dt} = e_i^{(a)} \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} Y^A$$

In the absence of corners, every infinitesimal deformation is therefore determined, up to initial data, by the knowledge of a vertical vector field along  $\hat{\gamma}$ , through the equation

$$(1.26) \quad X^a(t) = X^a(t_0) + \int_{t_0}^t e_i^{(a)} \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} Y^A d\tau$$

**1.3. Extremals.** In this Section we review the stationarity conditions for a given variational problem. Once again, the reader is referred to [1] for the full argumentation.

(i) Let  $\mathcal{L} \in \mathcal{F}(\mathcal{A})$  denote a differentiable function on the manifold  $\mathcal{A}$ , henceforth called the *Lagrangian*. Constrained calculus of variations deals with the study of the extremals of the functional  $\mathcal{I}[\gamma] := \int_{\hat{\gamma}} \mathcal{L} dt$  among all admissible sections  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  connecting two given configurations. We shall refer to this as *the control problem*.

It may be showed that, under suitable regularity assumptions, this is equivalent to a free variational problem on the contact bundle  $\mathcal{C}(\mathcal{A})$ . More specifically, denoting by  $v : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{V}_{n+1}$  the natural projection, the control problem may be converted into to the study of the extremals of a suitable action functional defined on differentiable sections  $\tilde{\gamma} : [t_0, t_1] \rightarrow \mathcal{C}(\mathcal{A})$  whose projections  $v \cdot \tilde{\gamma} : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  connect the given configurations.

As a matter of fact, the contact geometry and the presence of the Lagrangian function provide two “natural” variational problems on  $\mathcal{C}(\mathcal{A})$ . The former, of a purely geometric nature, is entirely determined by the Liouville 1-form (1.10) through the action functional

$$(1.27) \quad \tilde{\mathcal{I}}_0[\tilde{\gamma}] = \int_{\tilde{\gamma}} \tilde{\Theta} = \int_{t_0}^{t_1} p_i \left( \frac{dq^i}{dt} - \psi^i \right) dt$$

The corresponding extremals  $\tilde{\gamma} : q^i = q^i(t), z^A = z^A(t), p_i = p_i(t)$  are locally described by the equations

$$(1.28a) \quad \frac{dq^i}{dt} = \psi^i(t, q^i, z^A)$$

$$(1.28b) \quad \frac{dp_i}{dt} + \frac{\partial \psi^k}{\partial q^i} p_k = 0$$

$$(1.28c) \quad p_i \frac{\partial \psi^i}{\partial z^A} = 0$$

Their role is clarified by the following observations:

- a section  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  is admissible if and only if the functional (1.27) admits at least one extremal  $\tilde{\gamma}$  projecting onto  $\gamma$  i.e. satisfying  $v \cdot \tilde{\gamma} = \gamma$ ;



- the totality of extremals  $\tilde{\gamma}$  projecting onto an admissible section  $\gamma$  form a finite dimensional vector space  $\wp(\gamma)$ , identical to the space of solutions  $p_i = p_i(t)$  of eqs. (1.28b, c) for fixed  $q^i(t), z^A(t)$ .

A section  $\gamma$  is called *normal* if  $\dim \wp(\gamma) = 0$ , abnormal  $\dim \wp(\gamma) \neq 0$ ; it is called *locally normal* if its restriction to any closed subinterval  $[t_0^*, t_1^*] \subseteq [t_0, t_1]$  is a normal arc, i.e. if and only if along any such subinterval eqs. (1.28b, c) admit the one trivial solution  $p_i(t) = 0$ .

An illustrative example of a normal evolution  $\gamma$  which happen to be abnormal when restricted to a subinterval  $[t_0^*, t_1^*] \subset [t_0, t_1]$  may be given by means of a bump function as follows.

**Example 1.1.** Let  $\mathcal{V}_{n+1} = \mathbb{R} \times E_3$ , referred to coordinates  $t, q^1, q^2, q^3$ , be the configuration space–time of an abstract system  $\mathfrak{B}$  subject to non–holonomic constraints. Suppose the imbedding  $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$  is locally expressed as

$$\dot{q}^1 = z^1, \quad \dot{q}^2 = z^2, \quad \dot{q}^3 = g(t) z^2$$

being  $g(t)$  a  $C^\infty$ –function defined as  $g(t) := -\frac{2t}{(t^2-1)^2} e^{\frac{1}{t^2-1}}$  for any  $|t| < 1$  and  $g(t) := 0$  otherwise.

Consider now the evolution  $\gamma$  consisting of the arc:

$$\gamma: \quad q^1 = vt^2, \quad q^2 = vt, \quad q^3 = vf(t) \quad t_0 \leq t \leq t_1, \quad t_0 < -1, \quad t_1 > 1$$

being

$$f(t) := \begin{cases} e^{\frac{1}{t^2-1}} & |t| < 1 \\ 0 & |t| \geq 1 \end{cases}$$

It may be easily checked that, when restricted to the subinterval  $[t_0, -1]$ , equations (1.28b, c) admit the family of solutions of the form

$$p_1(t) = 0, \quad p_2(t) = 0, \quad p_3(t) = \alpha, \quad \forall \alpha \in \mathbb{R}$$

thus entailing the abnormality of the restriction of  $\gamma$  to  $[t_0, -1]$ . Notwithstanding  $\gamma$  is normal, since no solution may be found along the whole of it, but the trivial one.

The latter variational problem is obtained lifting the Lagrangian  $\mathcal{L}$  to a 1–form  $\vartheta_{\mathcal{L}}$  over  $\mathcal{C}(\mathcal{A})$  according to the prescription:<sup>3</sup>

$$(1.29) \quad \vartheta_{\mathcal{L}} := \mathcal{L} dt + \tilde{\Theta} = (\mathcal{L} - p_i \psi^i) dt + p_i dq^i := -\mathcal{H} dt + p_i dq^i$$

The function  $\mathcal{H}(t, q^i, z^A, p_i) := p_i \psi^i - \mathcal{L}$  is commonly referred to as the *Pontryagin Hamiltonian*.

The extremals of the functional  $\tilde{\mathcal{I}}[\tilde{\gamma}] := \int_{\tilde{\gamma}} \vartheta_{\mathcal{L}}$  are locally described by the *Pontryagin equations*

$$(1.30) \quad \frac{dq^i}{dt} = \psi^i(t, q^i, z^A), \quad \frac{dp_i}{dt} + \frac{\partial \psi^k}{\partial q^i} p_k = \frac{\partial \mathcal{L}}{\partial q^i}, \quad p_i \frac{\partial \psi^i}{\partial z^A} = \frac{\partial \mathcal{L}}{\partial z^A}$$

The formal analogy between eqs. (1.30) and the ones arising from the solution of the original control problem allows to set up a relationship between the extremals of  $\tilde{\mathcal{I}}[\tilde{\gamma}]$  in  $\mathcal{C}(\mathcal{A})$  and those of the functional  $\mathcal{I}[\gamma] := \int_{\gamma} \mathcal{L} dt$  in  $\mathcal{V}_{n+1}$ .

A detailed analysis of this point can be found in [1]: for the present purposes, it is enough recalling

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<sup>3</sup>To simplify things, we are not distinguishing between covariant objects in  $\mathcal{A}$  and their pull–back in  $\mathcal{C}(\mathcal{A})$ , i.e. we are writing  $\psi^i$  for  $\zeta^*(\psi^i)$ ,  $\tilde{\omega}^i$  for  $\zeta^*(\tilde{\omega}^i)$  etc.

that every extremal  $\tilde{\gamma} : [t_0, t_1] \rightarrow \mathcal{C}(\mathcal{A})$  of  $\tilde{I}[\tilde{\gamma}]$  projects onto an extremal  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  of the control problem.

(ii) A point  $\zeta \in \mathcal{C}(\mathcal{A})$  is called *regular* if and only if the relation

$$p_i \frac{\partial \psi^i}{\partial z^A} = \frac{\partial \mathcal{L}}{\partial z^A} \quad \left( \Longleftrightarrow \frac{\partial \mathcal{H}}{\partial z^A} = 0 \right)$$

can be solved for  $z^1, \dots, z^r$  in a neighborhood of  $\zeta$ , giving rise to local expressions of the form

$$(1.31) \quad z^A = z^A(t, q^1, \dots, q^n, p_1, \dots, p_n)$$

A sufficient condition for this to happen is the validity of the condition

$$(1.32) \quad \det \left( \frac{\partial^2 \mathcal{H}}{\partial z^A \partial z^B} \right)_{\zeta} \neq 0$$

In a neighborhood of each regular point, substituting eq. (1.31) into the first two eqs. (1.30) and introducing the notation  $\mathcal{H}(t, q^i, p_i) := \mathcal{H}(t, q^i, p_i, z^A(t, q^i, p_i))$ , Pontryagin equations can be written in Hamiltonian form as

$$(1.33) \quad \frac{dq^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q^i}$$

In this way, the actual constrained variational problem is locally reduced to a *free* Hamiltonian one.

## 2. THE SUFFICIENT CONDITIONS

**2.1. Lagrangians “adapted” to a given extremal curve.** A function  $f$  on a differentiable manifold  $M$  is said to be *critical* at a point  $x \in M$  if and only if its differential vanishes at  $x$ . Under the stated circumstance, the *hessian* of  $f$  at  $x$  determines a bilinear map  $(d^2 f)_x : T_x(M) \times T_x(M) \rightarrow \mathbb{R}$  and has therefore a tensorial character. Similar conclusions hold if  $f$  is critical at each point of a submanifold  $N \subset M$ , in which case we write  $(df)_N = 0$  and denote by  $(d^2 f)_N$  the hessian of  $f$  along  $N$ .

Denoting by  $\dot{f} := \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^k} \psi^k \in \mathcal{F}(\mathcal{A})$  the *symbolic time derivative* of any function  $f = f(t, q^i) \in \mathcal{F}(\mathcal{V}_{n+1})$ , we have then the following properties:

- if  $f$  is critical on an admissible section  $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$ , the function  $\dot{f}$  is critical on the lift  $\hat{\gamma}$  of  $\gamma$ , and satisfies  $\dot{f}|_{\hat{\gamma}} = 0$ ;

- under the same assumption, for any admissible deformation  $X : \mathbb{R} \rightarrow V(\gamma)$  the quadratic form  $\langle (d^2 f)_\gamma, X \otimes X \rangle = \left( \frac{\partial^2 f}{\partial q^i \partial q^k} \right)_\gamma X^i X^k$  satisfies the relation<sup>4</sup>

$$(2.1) \quad \frac{d}{dt} \langle (d^2 f)_\gamma, X \otimes X \rangle = \langle (d^2 \dot{f})_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle$$

The previous arguments are especially relevant in a variational context. For any  $f \in \mathcal{F}(\mathcal{V}_{n+1})$ , the action integrals  $\int_{\hat{\gamma}} \mathcal{L} dt$  and  $\int_{\hat{\gamma}} (\mathcal{L} - \dot{f}) dt$  have actually the same extremals with respect to fixed end-points deformations; in particular, every extremal  $\gamma$  yielding a minimum for the former integral does the same for the latter and conversely.

To explore the consequences of this fact, we consider once again the composite fibration  $v = \pi \cdot \zeta : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{V}_{n+1}$  described in §1.3 and recall that for each *normal* extremal  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  of the action integral  $\int_{\gamma} \mathcal{L} dt$  there exists a unique extremal  $\tilde{\gamma} : [t_0, t_1] \rightarrow \mathcal{C}(\mathcal{A})$  of the functional  $\int_{\tilde{\gamma}} \vartheta_{\mathcal{L}}$  projecting onto  $\gamma$ , i.e. satisfying  $\zeta \cdot \tilde{\gamma} = \hat{\gamma}$ , whence also  $v \cdot \tilde{\gamma} = \pi \cdot \hat{\gamma} = \gamma$ .

In coordinates, setting  $\tilde{\gamma} : q^i = q^i(t)$ ,  $z^A = z^A(t)$ ,  $p_i = p_i(t)$ , the determination of  $\tilde{\gamma}$  relies on Pontryagin equations (1.30). From these, it is easily seen that the evaluation of the functions  $p_i(t)$  is *not* invariant under gauge transformations  $\mathcal{L} \rightarrow \mathcal{L} - \dot{f}$ , but depends explicitly on the choice of the Lagrangian. This is easily understood observing that, for  $\mathcal{L}' = \mathcal{L} - \dot{f}$ , eq. (1.29) provides the identification

$$\vartheta_{\mathcal{L}} := p_i \omega^i + (\mathcal{L}' + \dot{f}) dt = \left( p_i - \frac{\partial f}{\partial q^i} \right) \omega^i + \mathcal{L}' dt + df$$

Because of that, the extremals of the functional  $\int_{\tilde{\gamma}} \vartheta_{\mathcal{L}'}$  differ from those of  $\int_{\tilde{\gamma}} \vartheta_{\mathcal{L}}$  by a translation

$$(2.2) \quad p_i(t) \rightarrow \bar{p}_i(t) = p_i(t) - \frac{\partial f}{\partial q^i}(t, q^i(t))$$

along the fibres of  $\mathcal{C}(\mathcal{A}) \xrightarrow{\zeta} \mathcal{A}$ , as it was to be expected on account of the gauge invariance of the projections  $\hat{\gamma} = \zeta \cdot \tilde{\gamma}$  and  $\gamma = v \cdot \tilde{\gamma}$ .

After these preliminaries, the idea is now to replace the original Lagrangian by a gauge equivalent one, satisfying the property of being *critical* along the section  $\hat{\gamma}$ .

To this end we state

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<sup>4</sup>As a check we observe that the condition  $(df)_\gamma = 0$  implies the identities

$$\begin{aligned} \left( \frac{\partial^2 \dot{f}}{\partial q^i \partial q^j} \right)_{\hat{\gamma}} &= \left[ \frac{\partial^3 f}{\partial q^i \partial q^j \partial t} + \frac{\partial^3 f}{\partial q^i \partial q^j \partial q^k} \psi^k + \frac{\partial^2 f}{\partial q^i \partial q^k} \frac{\partial \psi^k}{\partial q^j} + \frac{\partial^2 f}{\partial q^j \partial q^k} \frac{\partial \psi^k}{\partial q^i} \right]_{\hat{\gamma}} = \\ &= \frac{d}{dt} \left( \frac{\partial^2 f}{\partial q^i \partial q^j} \right)_\gamma + \left( \frac{\partial^2 f}{\partial q^i \partial q^k} \right)_\gamma \left( \frac{\partial \psi^k}{\partial q^j} \right)_{\hat{\gamma}} + \left( \frac{\partial^2 f}{\partial q^j \partial q^k} \right)_\gamma \left( \frac{\partial \psi^k}{\partial q^i} \right)_{\hat{\gamma}} \\ \left( \frac{\partial^2 \dot{f}}{\partial q^i \partial z^A} \right)_{\hat{\gamma}} &= \left( \frac{\partial^2 f}{\partial q^i \partial q^k} \right)_\gamma \left( \frac{\partial \psi^k}{\partial z^A} \right)_{\hat{\gamma}}, \quad \left( \frac{\partial^2 \dot{f}}{\partial z^A \partial z^B} \right)_{\hat{\gamma}} = 0 \end{aligned}$$

Eq. (2.1) is then easily obtained by direct computation, expressing the derivatives  $\frac{dX^i}{dt}$  in terms of the components  $X^i, \Gamma^A$  through the variational equation (1.14b).

**Definition 2.1.** *Given a normal extremal  $\gamma$ , a function  $S \in \mathcal{F}(\mathcal{V}_{n+1})$  is said to be adapted to  $\gamma$  if and only if its pull-back on  $\mathcal{C}(\mathcal{A})$  satisfies the condition<sup>5</sup>*

$$(2.3) \quad (dS)_{\hat{\gamma}} = (\vartheta \mathcal{L})_{\hat{\gamma}}$$

By a little abuse of language, whenever a function  $S \in \mathcal{F}(\mathcal{V}_{n+1})$  is adapted to  $\gamma$ , the same terminology will be used to denote the Lagrangian function  $\mathcal{L}' = \mathcal{L} - \dot{S}$ .

**Theorem 2.1.** *Eq. (2.3) entails the relations*

$$(2.4) \quad (\mathcal{L} - \dot{S})_{\hat{\gamma}} = [d(\mathcal{L} - \dot{S})]_{\hat{\gamma}} = 0$$

*Proof.* In coordinates, a possible local solution of eq.(2.3) is easily recognized to be

$$(2.5) \quad S_0(t, q^i) = p_i(t)(q^i - q^i(t)) + \int_0^t \mathcal{L}|_{\hat{\gamma}} dt$$

By linearity, the most general solution has therefore the local expression

$$(2.6) \quad S = S_0 + f(t, q^i)$$

being  $f(t, q^i)$  any function satisfying  $(df)_{\gamma} = 0$ .

On account of our previous remarks, in order to complete the proof we have only to check the validity of eqs. (2.4) for the function (2.5). To this end we evaluate

$$(2.7) \quad \dot{S}_0 = \frac{dp_i}{dt}(q^i - q^i(t)) + p_i \left( \psi^i - \frac{dq^i}{dt} \right) + \mathcal{L}|_{\hat{\gamma}}$$

Restricted to the curve  $\hat{\gamma}$ , eq. (2.7) yields back the relation  $(\mathcal{L} - \dot{S}_0)_{\hat{\gamma}} = 0$ . Moreover, eqs. (1.30), (2.7) provide the relations

$$\begin{aligned} \left( \frac{\partial \dot{S}_0}{\partial q^k} \right)_{\hat{\gamma}} &= \frac{dp_k}{dt} + p_i \left( \frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} = \left( \frac{\partial \mathcal{L}}{\partial q^k} \right)_{\hat{\gamma}} \\ \left( \frac{\partial \dot{S}_0}{\partial z^A} \right)_{\hat{\gamma}} &= p_i \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} = \left( \frac{\partial \mathcal{L}}{\partial z^A} \right)_{\hat{\gamma}} \end{aligned}$$

which, together with  $\frac{d}{dt} [(\mathcal{L} - \dot{S}_0)_{\hat{\gamma}}] = 0$ , imply the vanishing of the derivative  $Z(\mathcal{L} - S_0)$   $\forall Z \in T_{\hat{\gamma}(t)}(\mathcal{A})$ , and therefore the vanishing of  $[d(\mathcal{L} - \dot{S})]_{\hat{\gamma}}$ .  $\square$

For any  $S$  adapted to  $\gamma$ , let us now replace the original Lagrangian  $\mathcal{L}$  by the difference  $\mathcal{L}' = \mathcal{L} - \dot{S}$ . As already pointed out, both functions  $\mathcal{L}$  and  $\mathcal{L}'$  are totally equivalent for the study of the extremals in  $\mathcal{V}_{n+1}$ , as well as for their classification.

At the same time, by eq. (2.2), the extremals of the functionals  $\int_{\hat{\gamma}} \vartheta \mathcal{L}$  and  $\int_{\hat{\gamma}} \vartheta \mathcal{L}'$  differ by a translation  $p_i \rightarrow \bar{p}_i = p_i - \frac{\partial S}{\partial q^i}$  along the fibers of  $\mathcal{C}(\mathcal{A})$ . From this, taking eqs. (2.5), (2.6) into account it is easily seen that the unique extremal of  $\int_{\hat{\gamma}} \vartheta \mathcal{L}'$  projecting onto  $\gamma$  satisfies the local equation  $\bar{p}_i(t) = 0$ .

In view of eq. (2.6), the Lagrangian  $\mathcal{L}'$  is not unique, but is defined up to a *restricted gauge transformation*  $\mathcal{L}' \rightarrow \mathcal{L}' + \dot{f}$ , with  $(df)_{\gamma} = 0$ . In any case, for each choice of  $S$ , Theorem 2.1 ensures the tensorial character of the hessian  $(d^2 \mathcal{L}')_{\hat{\gamma}}$ .

<sup>5</sup>As usual, we are not distinguishing between functions on  $\mathcal{V}_{n+1}$  and their pull-back on  $\mathcal{C}(\mathcal{A})$ .

The local representation of  $(d^2 \mathcal{L}')_{\hat{\gamma}}$  is most easily expressed in non-holonomic bases. Setting  $\tilde{\omega}^i = (dq^i - \psi^i dt)_{\hat{\gamma}} = \omega^i|_{\hat{\gamma}}$ ,  $\tilde{\nu}^A = (dz^A - \frac{dz^A}{dt} dt)_{\hat{\gamma}}$ , a straightforward calculation yields the result

$$(2.8) \quad (d^2 \mathcal{L}')_{\hat{\gamma}} = \left[ \frac{\partial^2 \mathcal{L}'}{\partial q^r \partial q^k} \right]_{\hat{\gamma}} \tilde{\omega}^r \otimes \tilde{\omega}^k + 2 \left[ \frac{\partial^2 \mathcal{L}'}{\partial z^A \partial q^k} \right]_{\hat{\gamma}} \tilde{\nu}^A \odot \tilde{\omega}^k + \left[ \frac{\partial^2 \mathcal{L}'}{\partial z^A \partial z^B} \right]_{\hat{\gamma}} \tilde{\nu}^A \otimes \tilde{\nu}^B$$

$\odot$  denoting the symmetrized tensor product.

Concerning eq. (2.8), we observe the following facts:

- the components  $G_{AB} := \left[ \frac{\partial^2 \mathcal{L}'}{\partial z^A \partial z^B} \right]_{\hat{\gamma}}$  are invariant under arbitrary restricted gauge transformations, and may therefore be evaluated arbitrarily choosing the function  $S$  within the class of solutions of eqs. (2.3). Making use of the ansatz (2.5) we obtain the representation

$$G_{AB} = \left[ \frac{\partial^2 (\mathcal{L} - \dot{S})}{\partial z^A \partial z^B} \right]_{\hat{\gamma}} = \left[ \frac{\partial^2 \mathcal{L}}{\partial z^A \partial z^B} \right]_{\hat{\gamma}} - p_i(t) \left[ \frac{\partial^2 \psi^i}{\partial z^A \partial z^B} \right]_{\hat{\gamma}}$$

written more conveniently as

$$(2.9) \quad G_{AB} = - \left[ \frac{\partial^2 K}{\partial z^A \partial z^B} \right]_{\hat{\gamma}}$$

with  $K := p_i(t) \psi^i(t, q^i, z^A) - \mathcal{L}(t, q^i, z^A)$ , henceforth called the *restricted Pontryagin Hamiltonian*.

In view of the identification  $\left[ \frac{\partial^2 K}{\partial z^A \partial z^B} \right]_{\hat{\gamma}(t)} = \left[ \frac{\partial^2 \mathcal{H}}{\partial z^A \partial z^B} \right]_{\hat{\gamma}(t)}$ , the matrix (2.9) is automatically non-singular along any *regular* extremal.

- whenever  $\det G_{AB} \neq 0$ , the hessian (2.8) determines an *infinitesimal control* along  $\hat{\gamma}$ , namely a linear section  $h : V(\gamma) \rightarrow A(\hat{\gamma})$ , uniquely defined by the condition

$$(2.10a) \quad \left\langle (d^2 \mathcal{L}')_{\hat{\gamma}}, h(X) \otimes Y \right\rangle = 0 \quad \forall X \in V(\gamma), Y \in V(\hat{\gamma})$$

In view of eqs. (1.19), (2.8), the requirement (2.10a) is locally expressed by the relations

$$(2.10b) \quad \left\langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \tilde{\partial}_i \otimes \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}} \right\rangle = \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial z^A} \right)_{\hat{\gamma}} + G_{AB} h_i^B = 0$$

Under the assumption  $\det G_{AB} \neq 0$ , these may be solved for the components  $h_i^B$ , providing the representation

$$h_i^A = -G^{AB} \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial z^B} \right)_{\hat{\gamma}}$$

whence also

$$(2.11) \quad \tilde{\partial}_i := h \left[ \left( \frac{\partial}{\partial q^i} \right)_{\gamma} \right] = \left( \frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} - G^{AB} \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial z^B} \right)_{\hat{\gamma}} \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$$

with  $G^{AB} G_{BC} = \delta^A_C$ .

On account of eq. (2.10b), for any pair of vector fields  $U = U^i \tilde{\partial}_i + U^A \frac{\partial}{\partial z^A}$ ,  $W = W^i \tilde{\partial}_i + W^A \frac{\partial}{\partial z^A}$  in  $A(\hat{\gamma})$ , we have the relation

$$(2.12) \quad \left\langle (d^2 \mathcal{L}')_{\hat{\gamma}}, U \otimes W \right\rangle = N_{ij} U^i W^j + G_{AB} U^A W^B$$

with

$$(2.13) \quad N_{kr} := \langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \tilde{\partial}_k \otimes \tilde{\partial}_r \rangle = \left( \frac{\partial^2 \mathcal{L}'}{\partial q^k \partial q^r} \right)_{\hat{\gamma}} - G^{AB} \left( \frac{\partial^2 \mathcal{L}'}{\partial q^k \partial z^A} \right)_{\hat{\gamma}} \left( \frac{\partial^2 \mathcal{L}'}{\partial q^r \partial z^B} \right)_{\hat{\gamma}}$$

The absolute time derivative along  $\gamma$  induced by  $h$  will be denoted by  $\frac{D}{Dt}$ . The expression (1.22) for the temporal connection coefficients takes now the form

$$(2.14) \quad \tau_k^i := -\tilde{\partial}_k(\psi^i) = -\left( \frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} + G^{AB} \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \left( \frac{\partial^2 \mathcal{L}'}{\partial q^k \partial z^B} \right)_{\hat{\gamma}}$$

All properties stated in eqs. (1.23), (1.24) will be freely used in the following.

Unlike the components  $G_{AB}$ , the full hessian (2.8) and therefore also the associated infinitesimal control (if at all) and the corresponding time derivative are *not* gauge invariant, but depend on the specific choice of the Lagrangian  $\mathcal{L}'$ .

To set things straight, when working in a local chart  $(U, h)$  containing  $\hat{\gamma}$  we shall stick to the ansatz (2.5), thereby adopting the expression

$$(2.15) \quad \mathcal{L}' = \mathcal{L} - \frac{dp_i}{dt} (q^i - q^i(t)) - p_i(t) \left( \psi^i - \frac{dq^i}{dt} \right) - \mathcal{L}|_{\hat{\gamma}}$$

For later use, before moving on to the analysis of the second variation of the action functional, we eventually observe that, whenever  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  is a *normal* extremal, the use of an adapted Lagrangian provides a canonical splitting of the tangent space  $T(\mathcal{C}(\mathcal{A}))$ .

As already pointed out, the definition of the (unique) extremal curve  $\tilde{\gamma} : [t_0, t_1] \rightarrow \mathcal{C}(\mathcal{A})$  of the functional  $\int_{\tilde{\gamma}} \vartheta_{\mathcal{L}'}$  projecting onto  $\gamma$  is independent of the specific choice of  $\mathcal{L}'$  within the stated class of Lagrangians. In particular, the local representation of  $\tilde{\gamma}$  satisfies the condition  $p_i(t) \equiv 0$ .

Moreover, in view of the nature of  $\mathcal{C}(\mathcal{A}) \xrightarrow{\zeta} \mathcal{A}$  as a vector bundle over  $\mathcal{A}$ , each local section  $\mathcal{O} : \mathcal{A} \rightarrow \mathcal{C}(\mathcal{A})$  given by  $p_i(t, q^i, z^A) = 0$  has an invariant geometrical meaning. For each  $\sigma \in \mathcal{O}(\mathcal{A})$ , every vector  $\tilde{Z} \in T_\sigma(\mathcal{C}(\mathcal{A}))$  may therefore be split into a “horizontal” part  $\tilde{Z}^h := \mathcal{O}_* \zeta_*(\tilde{Z})$ , tangent to the submanifold  $\mathcal{O}(\mathcal{A})$ , and a “vertical” part  $\tilde{Z}^v$ , tangent to the fiber  $\zeta^{-1}(\zeta(\sigma))$ . In coordinates, we have the explicit representation

$$\tilde{Z}^v = \left\langle \tilde{Z}, (dp_i)_\sigma \right\rangle \left( \frac{\partial}{\partial p_i} \right)_\sigma, \quad \tilde{Z}^h = \tilde{Z} - \tilde{Z}^v$$

The previous algorithm interacts with another intrinsic attribute of the manifold  $\mathcal{C}(\mathcal{A})$ , represented by the Liouville 1-form (1.10). In this way, to each vector  $\tilde{Z} \in T_\sigma(\mathcal{C}(\mathcal{A}))$  we may associate a 1-form in  $T_\sigma^*(\mathcal{C}(\mathcal{A}))$  according to the prescription

$$(2.16) \quad \tilde{Z} \rightarrow \tilde{Z}^v \rightarrow \tilde{Z}^v \lrcorner (d\tilde{\Theta})|_\sigma = \left\langle \tilde{Z}, (dp_i)_\sigma \right\rangle \left( \frac{\partial}{\partial p_i} \lrcorner d\tilde{\Theta} \right)_\sigma = \left\langle \tilde{Z}, (dp_i)_\sigma \right\rangle \omega_\sigma^i$$

At last, we observe that every element of  $T_\sigma^*(\mathcal{C}(\mathcal{A}))$  generated by the process (2.16) may be uniquely expressed as the pull-back of a 1-form  $\lambda \in T_{v(\sigma)}^*(\mathcal{V}_{n+1})$ .

Collecting all results and recalling the definition of virtual 1-form introduced in §1.2 we conclude that the lift algorithm  $\gamma \rightarrow \tilde{\gamma}$  described above determines a bijective correspondence between vector fields  $\tilde{Z}$  along  $\tilde{\gamma}$  and pairs  $(\hat{Z}, \lambda)$ , in which  $\hat{Z} = \zeta_*(\tilde{Z})$  is a vector field along  $\hat{\gamma} = \zeta \cdot \tilde{\gamma}$ , and  $\lambda = \langle \tilde{Z}, (dp_i)_\sigma \rangle \hat{\omega}_\sigma^i$  is a virtual 1-form along  $\gamma$ .

**2.2. The second variation of the action functional.** Let  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  be a normal (not necessarily regular) extremal of the action functional  $\int_{\hat{\gamma}} \mathcal{L} dt$ . For any admissible deformation  $\gamma_\xi$  of  $\gamma$  we denote by  $\hat{\gamma}_\xi : q^i = \varphi^i(t, \xi)$ ,  $z^A = \zeta^A(t, \xi)$  the lift of  $\gamma_\xi$  to the velocity space  $\mathcal{A}$ , by  $\hat{X} = X^i \left( \frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + \Gamma^A \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$  the infinitesimal deformation tangent to  $\hat{\gamma}_\xi$  and by  $\mathcal{I}(\xi) := \int_{\hat{\gamma}_\xi} \mathcal{L} dt$  the value of the action integral along each path  $\hat{\gamma}_\xi$ .

By the very definition of extremal we have then  $\frac{d\mathcal{I}}{d\xi} \Big|_{\xi=0} = 0$  for all admissible deformations  $\gamma_\xi$  with fixed end-points. We now focus on the evaluation of the second derivative  $\frac{d^2\mathcal{I}}{d\xi^2} \Big|_{\xi=0}$ , commonly referred to as the *second variation* of the action functional at  $\gamma$ .

The algorithm is greatly simplified replacing  $\mathcal{L}$  by the difference  $\mathcal{L}' = \mathcal{L} - \dot{S}$ , being  $S \in \mathcal{F}(\mathcal{V}_{n+1})$  any function adapted to  $\gamma$  in the sense of Definition 2.1. By elementary calculus we have in fact the identity

$$\mathcal{I}(\xi) := \int_{\hat{\gamma}_\xi} (\mathcal{L}' + \dot{S}) dt = \int_{t_0}^{t_1} \mathcal{L}'(t, \varphi^i(t, \xi), \zeta^A(t, \xi)) dt + \text{const.}$$

In view of this as well as of the relation  $(d\mathcal{L}')_{\hat{\gamma}} = 0$ , the condition  $\frac{d\mathcal{I}}{d\xi} \Big|_{\xi=0} = 0$  is identically satisfied. As far as the second variation is concerned, a simple calculation yields the result

$$(2.17) \quad \frac{d^2\mathcal{I}}{d\xi^2} \Big|_{\xi=0} = \int_{t_0}^{t_1} \left[ \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial q^j} \right)_{\hat{\gamma}} X^i X^j + 2 \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial z^A} \right)_{\hat{\gamma}} X^i \Gamma^A + \left( \frac{\partial^2 \mathcal{L}'}{\partial z^A \partial z^B} \right)_{\hat{\gamma}} \Gamma^A \Gamma^B \right] dt = \int_{t_0}^{t_1} \langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle dt$$

As already pointed out, the hessian  $(d^2 \mathcal{L}')_{\hat{\gamma}}$  is not a gauge invariant object. More specifically, according to eq. (2.1), under a restricted gauge transformation  $\mathcal{L}' \rightarrow \mathcal{L}' + \dot{f}$ , the quadratic form (2.21) undergoes the transformation law

$$(2.18) \quad \langle [d^2(\mathcal{L}' + \dot{f})]_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle = \langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle + \frac{d}{dt} \langle (d^2 f)_\gamma, X \otimes X \rangle$$

As it was to be expected, eq. (2.18) shows that, unlike the integrand, the *integral* involved in the representation (2.17) of the second variation has a gauge-invariant behaviour, at least within the class of fixed end-points deformations.

As far as the second term at the right-hand-side of eq. (2.18) is concerned we notice that, due to the requirement  $(df)_\gamma = 0$ , the hessian  $(d^2 f)_\gamma$  determines a virtual tensor field along  $\gamma$ , expressed in components as<sup>6</sup>

$$(2.19) \quad (d^2 f)_\gamma = \left( \frac{\partial^2 f}{\partial q^i \partial q^j} \right)_\gamma \hat{\omega}^i \otimes \hat{\omega}^j$$

Conversely, for any symmetric virtual tensor  $C = C_{ij}(t) \hat{\omega}^i \otimes \hat{\omega}^j$  along  $\gamma$ , there exists at least a function  $f \in \mathcal{F}(\mathcal{V}_{n+1})$  defined in a neighborhood of  $\gamma$  and satisfying  $(df)_\gamma = 0$ ,  $(d^2 f)_\gamma = C$ .

The effect of the restricted gauge group on the representation (2.17) is therefore reflected into the fact that the integrand at the right-hand-side is defined up to an arbitrary transformation of

<sup>6</sup>See § 2.2 for notation and terminology.

the form

$$(2.20) \quad \langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle \longrightarrow \langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle + \frac{d}{dt} \langle C, X \otimes X \rangle$$

being  $C$  any symmetric virtual tensor field along  $\gamma$ .

The above arguments are further enhanced whenever  $\gamma$  is a *regular* extremal. First of all, introducing the horizontal basis (2.11) associated with the hessian  $(d^2 \mathcal{L}')_{\hat{\gamma}}$  and expressing  $\hat{X}$  in components as  $\hat{X} = X^i \tilde{\partial}_i + Y^A \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$ , eq. (2.12) provides the identification

$$(2.21) \quad \langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle = N_{kr} X^k X^r + G_{AB} Y^A Y^B$$

In view of that, denoting by  $\frac{D}{Dt}$  the absolute time derivative associated with the infinitesimal control (2.11) and recalling the representation (1.25a) of the variational equation in the basis  $\tilde{\partial}_i, \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$ , we conclude that, for any symmetric virtual tensor field  $C = C(t)$  along  $\gamma$ , there exists a restricted gauge transformation yielding the representation

$$(2.22) \quad \begin{aligned} \langle [d^2(\mathcal{L}' + \dot{f})]_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle &= \langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle + \frac{d}{dt} \langle C, X \otimes X \rangle = \\ &= \left( N_{ij} + \frac{DC_{ij}}{Dt} \right) X^i X^j + 2 C_{ij} \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} X^j Y^A + G_{AB} Y^A Y^B \end{aligned}$$

with the components  $\frac{DC_{ij}}{Dt}$  expressed in terms of the ordinary derivatives  $\frac{dC_{ij}}{dt}$  and of the temporal connection coefficients  $\tau_i^k = -\tilde{\partial}_i(\psi^k)$  by eq. (1.23).

On this basis we can state

**Theorem 2.2.** *Let  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  be a normal extremal. Then, if the matrix  $G_{AB}(t)$  is non singular at a point  $t^* \in (t_0, t_1)$ , there exist  $\varepsilon > 0$  and a restricted gauge transformation  $\mathcal{L}' \rightarrow \mathcal{L}' + \dot{f}$  such that the hessian  $[d^2(\mathcal{L}' + \dot{f})]_{\hat{\gamma}(t)}$  has algebraic rank equal to  $r$  for  $t \in (t^* - \varepsilon, t^* + \varepsilon)$ .*

*Proof.* By continuity, there exists an interval  $[c, d] \ni t^*$  where  $\det G_{AB} \neq 0$ . We focus on that interval, and apply eq. (2.22) to the arc  $\gamma([c, d])$ . Setting

$$\tilde{Y}^A := Y^A + G^{AB} C_{ir} \left( \frac{\partial \psi^r}{\partial z^B} \right)_{\hat{\gamma}} X^i$$

and taking the symmetry of  $C_{ij}$  into account, eq. (2.22) may be written as

$$\begin{aligned} \langle [d^2(\mathcal{L}' + \dot{f})]_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle &= \\ &= \left[ N_{ij} + \frac{DC_{ij}}{Dt} - G^{AB} \left( \frac{\partial \psi^r}{\partial z^A} \right)_{\hat{\gamma}} \left( \frac{\partial \psi^s}{\partial z^B} \right)_{\hat{\gamma}} C_{ir} C_{sj} \right] X^i X^j + G_{AB} \tilde{Y}^A \tilde{Y}^B \end{aligned}$$

The thesis is therefore established as soon as we prove that the Riccati-type differential equation

$$(2.23) \quad \frac{DC_{ij}}{Dt} - G^{AB} \left( \frac{\partial \psi^r}{\partial z^A} \right)_{\hat{\gamma}} \left( \frac{\partial \psi^s}{\partial z^B} \right)_{\hat{\gamma}} C_{ir} C_{sj} + N_{ij} = 0$$

admits at least one symmetric solution  $C_{ij} = C_{ij}(t)$  in a neighborhood of  $t = t^*$ .

To this end, we set

$$(2.24) \quad M^{rs} := G^{AB} \left( \frac{\partial \psi^r}{\partial z^A} \right)_{\hat{\gamma}} \left( \frac{\partial \psi^s}{\partial z^B} \right)_{\hat{\gamma}}$$



and denote by  $C_{ij}^{(S)}$  and  $C_{ij}^{(A)}$  respectively the symmetric and antisymmetric part of  $C_{ij}$ . Due to the symmetry of  $M^{ij}$  and  $N_{ij}$ , eq. (2.23) then splits into the system

$$(2.25a) \quad \frac{DC_{ij}^{(S)}}{Dt} - M^{rs} \left[ C_{ir}^{(S)} C_{sj}^{(S)} + C_{ir}^{(A)} C_{sj}^{(A)} \right] + N_{ij} = 0$$

$$(2.25b) \quad \frac{DC_{ij}^{(A)}}{Dt} + M^{rs} C_{ir}^{(S)} C_{sj}^{(A)} + M^{rs} C_{ir}^{(A)} C_{sj}^{(S)} = 0$$

Being the second equation linear and homogeneous in  $C_{ij}^{(A)}$ , by Cauchy theorem we conclude that, if we choose  $C_{ij}$  symmetric at  $t = t^*$ , there exists  $\varepsilon > 0$  such that the solution of the Cauchy problem for eq. (2.23) exists and is symmetric for  $|t - t^*| < \varepsilon$ .  $\square$

In view of Theorem 2.2, whenever  $\det G_{AB}(t^*) \neq 0$ , by a proper choice of the gauge around the point  $\gamma(t^*)$ , the quadratic polynomial (2.18) can be reduced to the canonical form

$$(2.26) \quad \left\langle [d^2(\mathcal{L}' + \dot{f})]_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \right\rangle = G_{AB} \tilde{Y}^A \tilde{Y}^B = - \left( \frac{\partial^2 K}{\partial z^A \partial z^B} \right)_{\hat{\gamma}} \tilde{Y}^A \tilde{Y}^B$$

in a neighborhood of  $t^*$ ,  $K$  denoting the restricted Pontryagin Hamiltonian.

Unfortunately, the purely *local* validity of eq. (2.26) makes it unsuited to the study of the second variation (2.17), which involves an integration over the whole interval  $[t_0, t_1]$ . We shall return on this point in § 2.3. At present, we concentrate on the role of Theorem 2.2 in the identification of *necessary* conditions for a regular extremal  $\gamma$  to yield a (relative) minimum for the action functional.

In this connection, a preliminary result is provided by the following

**Corollary 2.1.** *Under the same assumptions as in Theorem 2.2, given any vertical vector field  $V = V^A \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$  along  $\hat{\gamma}$  with compact support  $[a, b] \subset (t^* - \varepsilon, t^* + \varepsilon)$ , there exist a differentiable function  $g = g(t)$  not identically zero on  $[a, b]$  and an infinitesimal deformation  $\hat{X} = X^i \left( \frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + Y^A \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}$  with support contained in  $[a, b]$  satisfying the relation*

$$Y^A + G^{AB} C_{rs} \left( \frac{\partial \psi^s}{\partial z^B} \right)_{\hat{\gamma}} X^r = g V^A$$

*Proof.* Using the variational equation in the form (1.25), the required conditions are summarized into the pair of relations

$$(2.27a) \quad \frac{DX^i}{Dt} \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \left[ g V^A + G^{AB} C_{rs} \left( \frac{\partial \psi^s}{\partial z^B} \right)_{\hat{\gamma}} X^r \right]$$

$$(2.27b) \quad X^i(a) = X^i(b) = 0$$

For any choice of  $g(t)$ , eq. (2.27) is a first order linear differential equation for the unknowns  $X^i(t)$ ; integrating it with initial data  $X^i(a) = 0$  yields the solution

$$X^i(t) = W^i_k(t) \int_a^t (W^{-1})^k_r \left( \frac{\partial \psi^r}{\partial z^A} \right)_{\hat{\gamma}} g V^A d\xi$$

being  $W^i_k$  the *Wronskian* of the equation. In order to ensure  $X^i(b) = 0$  it is then sufficient to choose  $g(t)$  within the (infinite-dimensional) vector space of differentiable functions over  $(t^* - \varepsilon, t^* + \varepsilon)$

satisfying the conditions

$$\int_a^b (W^{-1})^k_r \left( \frac{\partial \psi^r}{\partial z^A} \right)_{\hat{\gamma}} g V^A d\xi = 0, \quad k = 1 \dots n$$

□

**Corollary 2.2.** *A necessary condition for a normal extremal  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  to yield a minimum for the action functional is that the matrix  $G_{AB}(t)$  be positive semidefinite at all  $t \in [t_0, t_1]$ .*

*Proof.* Suppose that  $G_{AB}$  is not positive semidefinite at some  $t^* \in [t_0, t_1]$ . Depending on the value of  $\det G_{AB}(t^*)$  we have then two possible alternatives:

i) if  $\det G_{AB}(t^*) \neq 0$ , on account of Theorem 2.2 there exist a restricted gauge transformation  $\mathcal{L}' \rightarrow \mathcal{L}' + \dot{f}$  such that a representation like (2.26) holds in a neighborhood  $(t^* - \varepsilon, t^* + \varepsilon)$ .

Then, given any vertical vector field  $V$  with support contained in  $(t^* - \varepsilon, t^* + \varepsilon)$  and satisfying  $G_{AB} V^A V^B < 0$  (for instance, the eigenvector corresponding to the negative eigenvalue of  $G_{AB}$  in  $(t^* - \varepsilon, t^* + \varepsilon)$ , multiplied by a suitable function with compact support), Corollary 2.1 implies the existence of at least one infinitesimal deformation  $\hat{X}$  satisfying

$$\left. \frac{d^2 \mathcal{I}}{d\xi^2} \right|_{\xi=0} = \int_{t_0}^{t_1} \langle [d^2(\mathcal{L}' + \dot{f})]_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle dt = \int_a^b g^2 G_{AB} V^A V^B dt < 0$$

Therefore,  $\gamma$  does not provide a minimum for the action functional.

ii) if  $\det G_{AB}(t^*) = 0$ , choose  $\varepsilon > 0$  in such a way that

- $-\varepsilon$  is not a root of the secular equation  $\det(G_{AB} - \lambda \delta_{AB}) = 0$ ;
- at least one root of the secular equation is smaller than  $-\varepsilon$ .

Let  $\mathcal{M} \in \mathcal{F}(\mathcal{A})$  be a differentiable function globally defined on  $\mathcal{A}$  and having local expression  $\mathcal{M} = \varepsilon \delta_{AB} (z^A - z^A(t))(z^B - z^B(t))$  in a neighborhood  $U$  of the point  $\hat{\gamma}(t^*)$ , where, as usually, we are writing  $z^A(t)$  for  $z^A(\hat{\gamma}(t))$ . Also, let  $[c, d] \ni t^*$  be a closed interval, satisfying  $\hat{\gamma}([c, d]) \subset U$ . Setting  $\mathcal{L}^* := \mathcal{L}' + \mathcal{M}$ , one can then easily verify the properties:

- a) the section  $\gamma : [c, d] \rightarrow \mathcal{V}_{n+1}$  is a normal extremal for the action integral  $\int_{\hat{\gamma}} L^* dt$ ;
- b) the matrix  $\left( \frac{\partial^2 L^*}{\partial z^A \partial z^B} \right)_{\hat{\gamma}(t^*)} = G_{AB} + \varepsilon \delta_{AB}$  is both non singular and non positive (semi)-definite.

In view of a) and b), the analysis developed in point i) ensures the existence of at least one infinitesimal deformation  $\hat{X}$  having support in  $[a, b] \subset [c, d]$  and satisfying

$$\int_c^d \langle (d^2 \mathcal{L}^*)_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle dt < 0$$

On the other hand, by construction, this implies also

$$\begin{aligned} \int_c^d \langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle dt &= \int_c^d \langle (d^2 \mathcal{L}^*)_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle dt + \\ &\quad - \varepsilon \int_c^d \delta_{AB} \langle dz^A, \hat{X} \rangle \langle dz^B, \hat{X} \rangle dt \leq \int_c^d \langle (d^2 \mathcal{L}^*)_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \rangle dt < 0 \end{aligned}$$

once again proving that  $\gamma$  does not yield a minimum for the action functional. □

**2.3. Regular extremals: the matrix Riccati equation.** From now onward we shall concentrate on the class of *regular* normal extremals. The role of regularity in the solution of the Pontryagin equations (1.30) — more specifically, in the conversion of these into a system of ordinary differential equations in Hamiltonian form for the unknowns  $q^i(t), p_i(t)$  — is well-known in the literature.<sup>7</sup> However, in the present context, regularity is meant as a merely *attribute* of a given extremal  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$ , ensuring the existence of an expression of the form (2.26) in a neighborhood of each  $t^* \in [t_0, t_1]$ .

On the other hand, as already pointed out, the purely local validity of eq. (2.26) is of little help in the evaluation of the second variation (2.17). In this connection, it ought to be established to what extent eq. (2.26) may be converted into a global result, valid over the whole interval  $[t_0, t_1]$ . On account of eqs. (2.23), (2.24), this entails to analyze the interval of existence of the solutions of the Riccati-like differential equation<sup>8</sup>

$$(2.28) \quad \frac{DC_{ij}}{Dt} - M^{rs} C_{ir} C_{sj} + N_{ij} = 0$$

The main difficulty comes, of course, from its non-linearity. To overcome this aspect, we introduce two auxiliary virtual tensors  $E_{ij}(t)$  and  $K^i_j(t)$  along  $\gamma$ , satisfying the transport laws

$$(2.29a) \quad \frac{DK^i_j}{Dt} = M^{ir} E_{rj}$$

$$(2.29b) \quad \frac{DE_{ij}}{Dt} = N_{ir} K^r_j$$

On any interval  $(a, b)$  on which  $\det K^i_j(t) \neq 0$ , the (generally non symmetric) tensor

$$(2.30) \quad C_{ij} = -E_{ir} (K^{-1})^r_j$$

is thus well-defined and fulfils the relation

$$-\frac{DE_{ip}}{Dt} = \frac{DC_{ir}}{Dt} K^r_p + C_{ir} \frac{DK^r_p}{Dt}$$

Substituting from eqs. (2.29a), (2.30) and multiplying by  $(K^{-1})^p_j$  the latter may be written in the form

$$(2.28') \quad -N_{ij} = \frac{DC_{ij}}{Dt} + C_{ir} M^{rs} E_{sp} (K^{-1})^p_j = \frac{DC_{ij}}{Dt} - C_{ir} M^{rs} C_{sj}$$

formally identical to eq. (2.28).

Needless to say, the symmetry property  $C_{ij} = C_{ji}$  is also needed in order for the tensor (2.30) to represent the hessian of a function  $f$  along  $\gamma$ . In view of eq. (2.28'), the antisymmetric part of  $C_{ij}$  obeys a linear homogeneous system of the form (2.25b) and therefore an argument similar to the one exploited in the proof of Theorem 2.2 shows that this aspect relies entirely on the choice of the initial data. Once again, by Cauchy theorem we conclude that if  $C_{ij}$  is symmetric at  $t = t_0$  (as it happens e.g. choosing  $E_{ir}(t_0) = 0$ ,  $K^r_j(t_0) = \delta^r_j$ ), it retains its symmetric character up to the first value  $t^* > t_0$  (if any) at which  $\det K^r_j(t^*) = 0$ .

<sup>7</sup>See e.g. [1].

<sup>8</sup>The regularity assumption is once again crucial in ensuring the *global* character of the absolute time derivative  $\frac{D}{Dt}$  induced by the hessian  $(d^2\mathcal{L}')_{\dot{\gamma}}$  along  $\gamma$ .

**Remark 2.1.** *The analysis of eqs. (2.28), (2.29a, b) is considerably simplified referring the virtual tensor algebra along  $\gamma$  to an  $h$ -transported basis  $\{e^{(a)}, e_{(a)}\}$  and recalling that, in this way, the components  $\frac{DT^a_{b\dots}}{Dt}$  of the absolute time derivative of a field  $T$  coincide with the ordinary derivatives  $\frac{dT^a_{b\dots}}{dt}$  (§ 2.2, eq. (1.24)). Eq. (2.28) reduces then to the ordinary matrix Riccati equation*

$$(2.31) \quad \frac{dC_{ab}}{dt} - M^{rs} C_{ar} C_{sb} + N_{ab} = 0$$

while eqs. (2.29a, b) take the simpler form

$$(2.32a) \quad \frac{dK^a_b}{dt} = M^{ac} E_{cb}$$

$$(2.32b) \quad \frac{dE_{ab}}{dt} = N_{ac} K^c_b$$

Collecting all results, we can now state

**Theorem 2.3** (Sufficient conditions). *Let  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  be a normal extremal for the action functional. Also, let  $\mathcal{H} = p_i \psi^i - \mathcal{L}$  denote the Pontryagin Hamiltonian associated with the given Lagrangian. Then, if the matrix*

$$G_{AB}(t) := -\left(\frac{\partial^2 K}{\partial z^A \partial z^B}\right)_{\hat{\gamma}} = -\left(\frac{\partial^2 \mathcal{H}}{\partial z^A \partial z^B}\right)_{\hat{\gamma}}$$

*is positive definite at each  $t \in [t_0, t_1]$  and if the system (2.29a, b) admits at least one solution  $E_{ij}(t), K^i_j(t)$  fulfilling the conditions*

- $E_{ir} (K^{-1})^r_j$  symmetric,
- $\det K^i_j \neq 0$  everywhere on  $[t_0, t_1]$ ,

*the section  $\gamma$  yields a weak local minimum of the action functional.*

*Proof.* The stated assumptions imply both the regularity of the extremal  $\gamma$  and the existence of a global solution of the Riccati-like equation (2.28) along  $\gamma$ , thus ensuring the validity of an expression like eq. (2.28) on the whole interval  $[t_0, t_1]$ . On account of eq. (2.17), if the matrix  $G_{AB}$  is positive definite on  $[t_0, t_1]$ , this provides the evaluation

$$\left.\frac{d^2 \mathcal{I}}{d\xi^2}\right|_{\xi=0} = \int_{t_0}^{t_1} \left\langle [d^2(\mathcal{L}' + \dot{f})]_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \right\rangle dt = \int_{t_0}^{t_1} G_{AB} \tilde{Y}^A \tilde{Y}^B dt > 0$$

for every non-zero admissible deformation  $\hat{X} : [t_0, t_1] \rightarrow A(\hat{\gamma})$ . □

A deeper insight into the meaning of the condition  $\det K^i_j \neq 0$  is provided by the study of the *Jacobi vector fields*, reviewed and adapted to the present geometrical context in § 3.

### 3. THE NECESSARY CONDITIONS

**3.1. Jacobi fields.** Given a regular normal extremal  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  of the action functional  $\int_{\hat{\gamma}} \mathcal{L} dt$ , we now focus our attention on the (unique) extremal  $\tilde{\gamma} : [t_0, t_1] \rightarrow \mathcal{C}(\mathcal{A})$  of the functional  $\int_{\tilde{\gamma}} \vartheta_{\mathcal{L}'}$  projecting onto  $\gamma$ . Also, as usual, we preserve the notation  $v = \pi \cdot \zeta : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{V}_{n+1}$  for the composite fibration  $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A} \rightarrow \mathcal{V}_{n+1}$ .

Let us now introduce a special class of deformations  $\tilde{\gamma}_\xi$  of  $\tilde{\gamma}$  in which every section  $\tilde{\gamma}_\xi : [t_0, t_1] \rightarrow \mathcal{C}(\mathcal{A})$  is itself an *extremal* of  $\int_{\tilde{\gamma}} \vartheta \mathcal{L}'$ .

In this way, the 1-parameter family  $\gamma_\xi := v \cdot \tilde{\gamma}_\xi$  is a deformation of the original section  $\gamma$ , consisting of extremals of the functional  $\int_{\tilde{\gamma}} \mathcal{L} dt$ . At this stage, we do not impose any restriction on the behaviour of the end-points  $\gamma_\xi(t_0)$ ,  $\gamma_\xi(t_1)$ . In coordinates, setting

$$(3.1) \quad \tilde{\gamma}_\xi : \quad q^i = \varphi^i(t, \xi), \quad z^A = \zeta^A(t, \xi), \quad p_i = \bar{\rho}_i(t, \xi)$$

our assumptions are summarized into the request that, for each value of  $\xi$ , the functions at the right-hand-side of eqs. (3.1) satisfy Pontryagin equations

$$(3.2a) \quad \frac{\partial \varphi^i}{\partial t} = \psi^i(t, \varphi^i, \zeta^A)$$

$$(3.2b) \quad \frac{\partial \bar{\rho}_i}{\partial t} + \frac{\partial \psi^k}{\partial q^i} \bar{\rho}_k = \frac{\partial \mathcal{L}'}{\partial q^i}$$

$$(3.2c) \quad \bar{\rho}_i \frac{\partial \psi^i}{\partial z^A} = \frac{\partial \mathcal{L}'}{\partial z^A}$$

As a check of inner consistency it is worth observing that, in view of the condition  $(d\mathcal{L}')_{\tilde{\gamma}} = 0$ , eqs. (3.2b, c) and the normality of  $\gamma$  yield back the relation  $\bar{\rho}_i(t, 0) = 0$ .

Strictly associated with  $\tilde{\gamma}_\xi$  is a corresponding *infinitesimal deformation*, locally expressed as  $\tilde{X} = X^i \left( \frac{\partial}{\partial q^i} \right)_{\tilde{\gamma}} + \Gamma^A \left( \frac{\partial}{\partial z^A} \right)_{\tilde{\gamma}} + \bar{\pi}_i \left( \frac{\partial}{\partial p_i} \right)_{\tilde{\gamma}}$ , with

$$(3.3) \quad X^i = \left( \frac{\partial \varphi^i}{\partial \xi} \right)_{\xi=0}, \quad \Gamma^A = \left( \frac{\partial \zeta^A}{\partial \xi} \right)_{\xi=0}, \quad \bar{\pi}_i = \left( \frac{\partial \bar{\rho}_i}{\partial \xi} \right)_{\xi=0}$$

Taking eqs. (3.2) and the relation  $\bar{\rho}_i(t, 0) = 0$  into account, it is easily seen that the components (3.3) satisfy the following system of differential-algebraic equations

$$(3.4a) \quad \frac{dX^i}{dt} = \left( \frac{\partial \psi^i}{\partial q^k} \right)_{\tilde{\gamma}} X^k + \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\tilde{\gamma}} \Gamma^A$$

$$(3.4b) \quad \frac{d\bar{\pi}_i}{dt} + \bar{\pi}_k \left( \frac{\partial \psi^k}{\partial q^i} \right)_{\tilde{\gamma}} = \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial q^k} \right)_{\tilde{\gamma}} X^k + \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial z^A} \right)_{\tilde{\gamma}} \Gamma^A$$

$$(3.4c) \quad \bar{\pi}_i \frac{\partial \psi^i}{\partial z^A} = \left( \frac{\partial^2 \mathcal{L}'}{\partial z^A \partial q^k} \right)_{\tilde{\gamma}} X^k + \left( \frac{\partial^2 \mathcal{L}'}{\partial z^A \partial z^B} \right)_{\tilde{\gamma}} \Gamma^B$$

Given any vector field  $\tilde{X}$  satisfying eqs. (3.4), its push-forward  $v_* \tilde{X}$  will be called a *Jacobi field* along  $\gamma$ . By definition, a Jacobi field  $X = X^i \left( \frac{\partial}{\partial q^i} \right)_\gamma$  is therefore the infinitesimal deformation tangent to a finite deformation  $\gamma_\xi$  consisting of a 1-parameter family of extremals of the functional  $\int_{\tilde{\gamma}} \mathcal{L} dt$ .

**Remark 3.1.** *The resemblance between eqs. (3.4) and Pontryagin ones (1.30) sticks out a mile. This aspect can be made explicit by replacing the imbedding (1.5a,b) with its linearized counterpart (1.13a,b), namely regarding the vector bundle  $V(\gamma)$  as the configuration space-time of an abstract system  $\mathfrak{B}'$  and the bundle  $A(\tilde{\gamma}) \rightarrow V(\gamma)$  as the associated space of admissible velocities. In this way, the admissible evolutions of  $\mathfrak{B}'$  are easily seen to be in bijective correspondence with the admissible infinitesimal deformations of  $\gamma$ .*

Referring  $V(\gamma)$  to coordinates  $t, u^i$  and  $A(\hat{\gamma})$  to coordinates  $t, u^i, v^A$  according to the prescriptions (1.12a,b), the imbedding  $i_* : A(\hat{\gamma}) \rightarrow j_1(V(\gamma))$  is locally expressed by eq. (1.13b), synthetically written as

$$(3.5) \quad \dot{u}^i = \left( \frac{\partial \psi^i}{\partial q^k} \right)_{\hat{\gamma}} u^k + \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} v^A := \Psi^i(t, u^i, v^A)$$

To complete the picture, we adopt the quadratic form

$$(3.6) \quad \mathfrak{L}(\hat{X}) := \left\langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \right\rangle$$

as a Lagrangian on  $A(\hat{\gamma})$ , and denote by  $\mathfrak{J}$  the functional assigning to each admissible section  $X : [t_0, t_1] \rightarrow V(\gamma)$  the action integral  $\mathfrak{J}[X] := \int_{\hat{X}} \mathfrak{L} dt$ .

In this way, for any finite deformation  $\gamma_\xi$  of  $\gamma$  tangent to  $X$ , eq. (2.17) provides the identification  $\mathfrak{J}[X] = \frac{1}{2} \frac{d^2 \mathcal{I}}{d\xi^2} \Big|_{\xi=0}$ .

In coordinates, eq. (3.6) takes the explicit form

$$\mathfrak{L}(t, u^i, v^A) = \frac{1}{2} \left[ \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial q^j} \right)_{\hat{\gamma}} u^i u^j + 2 \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial z^A} \right)_{\hat{\gamma}} u^i v^A + \left( \frac{\partial^2 \mathcal{L}'}{\partial z^A \partial z^B} \right)_{\hat{\gamma}} v^A v^B \right]$$

From the latter it may be easily verified that the equations (3.4) involved in the definition of the Jacobi fields are formally identical to the Pontryagin equations for the determination of the extremals of the functional  $\mathfrak{J}$  subject to the constraints (3.5).

Rather than pursuing the analogy outlined in the previous Remark, we work directly on the system (3.4). Recalling the discussion at the end of §2.1, the field  $\tilde{X}$  may be decomposed into the pair

$$\hat{X} = X^i \left( \frac{\partial}{\partial q^i} \right)_{\hat{\gamma}} + \Gamma^A \left( \frac{\partial}{\partial z^A} \right)_{\hat{\gamma}}, \quad \lambda = \bar{\pi}_i \hat{\omega}^i|_{\gamma}$$

in which  $\hat{X}$  is a vector field along  $\hat{\gamma}$  while  $\lambda$  is a virtual 1-form along  $\gamma$ . By a little abuse of language, this will be called a *Jacobi pair* belonging to  $X = v_* \tilde{X}$ .

Under the (crucial) hypothesis of regularity of  $\gamma$ , we now make use of the infinitesimal control  $h : V(\gamma) \rightarrow A(\hat{\gamma})$  induced by the Lagrangian  $\mathcal{L}'$  to express the field  $\hat{X}$  in terms of the Jacobi field  $X$  and of a vertical vector  $Y$  in the form  $\hat{X} = h(X) + Y = X^i \hat{\partial}_i + Y^A \frac{\partial}{\partial z^A}$ . On account of eqs. (2.11), this implies the relation

$$(3.7) \quad \Gamma^A = Y^A - G^{AB} \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial z^B} \right)_{\hat{\gamma}} X^i$$

Together with the identification  $G_{AB} = \left( \frac{\partial^2 \mathcal{L}'}{\partial z^A \partial z^B} \right)_{\hat{\gamma}}$ , the latter allows to cast eq. (3.4c) into the form

$$(3.8) \quad \bar{\pi}_i \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} = G_{AB} Y^B \implies Y^A = G^{AB} \bar{\pi}_i \left( \frac{\partial \psi^i}{\partial z^B} \right)_{\hat{\gamma}}$$

From this, substituting into equations (3.4b, c), recalling the definitions (2.13), (2.24) of the tensors  $N_{ij}$ ,  $M^{ij}$  and expressing everything in terms of the absolute time derivative, we eventually

obtain the following system of differential equations for the unknowns  $X^i(t), \bar{\pi}_i(t)$ :

$$(3.9a) \quad \frac{DX^i}{Dt} = G^{AB} \left( \frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} \left( \frac{\partial \psi^j}{\partial z^B} \right)_{\hat{\gamma}} \bar{\pi}_j = M^{ij} \bar{\pi}_j$$

$$(3.9b) \quad \frac{D\bar{\pi}_i}{Dt} = \left[ \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial q^j} \right)_{\hat{\gamma}} - G^{AB} \left( \frac{\partial^2 \mathcal{L}'}{\partial q^i \partial z^A} \right)_{\hat{\gamma}} \left( \frac{\partial^2 \mathcal{L}'}{\partial q^j \partial z^B} \right)_{\hat{\gamma}} \right] X^j = N_{ij} X^j$$

As the attentive reader will have noticed, these are formally identical to the linearized form of Riccati equation (2.29): a result that will prove to be fundamental in the sequel in order to determine the necessary and sufficient conditions for a weak local minimum.

**Remark 3.2.** *Keeping in line with Remark 3.1, if the virtual tensor algebra along  $\gamma$  is referred to an  $h$ -transported basis  $\{e^{(a)}, e_{(a)}\}$ , the set of  $2n$  differential equations (3.9) are written in the form*

$$(3.10) \quad \frac{dX^a}{dt} = \hat{M}^{ab} \bar{\pi}_b, \quad \frac{d\bar{\pi}_a}{dt} = \hat{N}_{ab} X^b$$

Once again, these are easily seen to represent the Hamilton equations for the Hamiltonian function

$$(3.11) \quad \mathfrak{H}(t, X^a, \bar{\pi}_b) = \bar{\pi}_a \Psi^a - \mathfrak{L} = \frac{1}{2} M^{ab} \bar{\pi}_a \bar{\pi}_b - \frac{1}{2} N_{ab} X^a X^b$$

The relationship between Jacobi fields and the second variation is made evident by the following

**Proposition 3.1.** *Given a Jacobi pair  $(\hat{X}, \lambda)$ , any infinitesimal deformation  $\hat{Z} = Z^i(\tilde{\partial}_i)_{\hat{\gamma}} + Z^A(\frac{\partial}{\partial z^A})_{\hat{\gamma}}$  satisfies the relation*

$$(3.12) \quad \left\langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{X} \otimes \hat{Z} \right\rangle = \frac{d(\bar{\pi}_i Z^i)}{dt}$$

*Proof.* Eqs. (1.25a) and (3.9b) provide the evaluation

$$\frac{d(\bar{\pi}_i Z^i)}{dt} = \frac{D(\bar{\pi}_i Z^i)}{Dt} = N_{ij} X^i Z^j + \bar{\pi}_i \frac{\partial \psi^i}{\partial z^A} Z^A = N_{ij} X^i Z^j + G_{AB} Y^B Z^A$$

The thesis immediately follows in the light of eq. (2.12).  $\square$

**Remark 3.3.** *Hitherto, our treatment of Jacobi fields has uniquely involved the adapted Lagrangian  $\mathcal{L}'$ . This choice was suggested both by consistency with the previous analysis and also by the simplified calculations. However, it is not at all necessary in order to cover the subject. The same considerations can be obviously done sticking on the original Lagrangian function. In this way, equations (3.4) are directly written in terms of the Pontryagin Hamiltonian  $\mathcal{H}$ , with the quantities  $\bar{\pi}_i$  replaced by  $\pi_i := (\frac{\partial \rho_i}{\partial \xi})_{\xi=0}$  and related to the previous ones by the relation*

$$\bar{\pi}_i(t) = \left( \frac{\partial \bar{\rho}_i(t, \xi)}{\partial \xi} \right)_{\xi=0} = \frac{\partial}{\partial \xi} \left( \rho_i(t, \xi) - \frac{\partial S}{\partial q^i} \right)_{\xi=0} = \pi_i(t) - \frac{\partial^2 S}{\partial q^i \partial q^j} X^j$$

*The argument is almost identical to the one developed so far and will be omitted.*

**3.2. Conjugate points.** The arguments of § 3.1 come helpful for establishing the necessary conditions for a given extremal to yield a local minimum for the action functional.

**Definition 3.1** (Conjugate point). *A point  $\gamma(\tau)$ ,  $\tau \in (t_0, t_1]$ , along a given extremal curve  $\gamma: [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  is said to be conjugate to  $\gamma(t_0)$  if and only if there exists a non-zero Jacobi field  $X: [t_0, t_1] \rightarrow V(\gamma)$  such that  $X(t_0) = X(\tau) = 0$ .*

The search for conjugate points can be performed by looking for a solution of equations (3.9) with  $X^i(t_0) = 0$  and  $\bar{\pi}_i(t_0)$  varying amongst all the possible values in  $\mathbb{R}^n$ . Because of the linearity of equations (3.9), their solution will depend on the initial data through a set of time-dependent matrices in the form

$$(3.13a) \quad X^i(t) = \mathcal{A}_j^i(t, t_0) X^j(t_0) + \mathcal{B}^{ij}(t, t_0) \bar{\pi}_j(t_0)$$

$$(3.13b) \quad \bar{\pi}_i(t) = \mathcal{C}_{ij}(t, t_0) X^j(t_0) + \mathcal{D}_i^j(t, t_0) \bar{\pi}_j(t_0)$$

with  $\mathcal{A}_j^i(t_0, t_0) = \mathcal{D}_i^j(t_0, t_0) = \delta_j^i$ ,  $\mathcal{B}^{ij}(t_0, t_0) = \mathcal{C}_{ij}(t_0, t_0) = 0$ .

Hence, by Definition 3.1, a point  $\gamma(\tau)$  is *conjugate* to  $\gamma(t_0)$  if and only if the map  $V^*(\gamma)|_{t_0} \rightarrow V(\gamma)|_t$  locally described by

$$(3.14) \quad X^i(t) = \mathcal{B}^{ij}(t, t_0) \bar{\pi}_j(t_0)$$

is injective for  $t \in (t_0, \tau)$  but not for  $t = \tau$ , i.e. if and only if the associated matrix satisfies  $\det B^{ij}(t, t_0) \neq 0$  for  $t_0 < t < \tau$ , and  $\det B^{ij}(\tau, t_0) = 0$ .

The link between conjugate points and the analysis of the second variation is clarified by the following generalization of a classical result of Bliss ([25]):

**Theorem 3.1.** *Consider an extremal curve  $\gamma: [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  and suppose there exists a value  $\tau \in (t_0, t_1)$  such that the point  $\gamma(\tau)$  is conjugate to  $\gamma(t_0)$ . Then, the quadratic form*

$$(3.15) \quad \int_{t_0}^{t_1} \left\langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \right\rangle dt$$

*is indefinite within the class of infinitesimal deformations  $X$  vanishing at the end-points.*

*Proof.* Let us define a symmetric bilinear functional  $(d^2 \mathcal{I})_{\hat{\gamma}}$  over  $A(\hat{\gamma})$  as

$$\left\langle (d^2 \mathcal{I})_{\hat{\gamma}}, \hat{V} \otimes \hat{W} \right\rangle := \int_{t_0}^{t_1} \left\langle (d^2 \mathcal{L}')_{\hat{\gamma}}, \hat{V} \otimes \hat{W} \right\rangle dt$$

for any  $\hat{V}, \hat{W}$  in  $A(\hat{\gamma})$ . Then, by a well-known result in the theory of quadratic forms, the thesis is proved as soon as we show that, in the presence of a point  $\gamma(\tau)$  conjugated to  $\gamma(t_0)$ , the kernel<sup>9</sup> of  $(d^2 \mathcal{I})_{\hat{\gamma}}$  does not coincide with the locus of zeroes of its associated quadratic form.

Under the stated hypothesis, there exists a non-zero Jacobi field  $J \in V(\gamma)$  such that  $J(t_0) = J(\tau) = 0$ . By means of this, we now define a continuous infinitesimal deformation vanishing

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<sup>9</sup>We recall that the kernel of the bilinear functional  $(d^2 \mathcal{I})_{\hat{\gamma}}: A(\hat{\gamma}) \times A(\hat{\gamma}) \rightarrow \mathbb{R}$  is defined as the set  $\ker(d^2 \mathcal{I})_{\hat{\gamma}} = \{\hat{V} \mid \hat{V} \in A(\hat{\gamma}), \langle (d^2 \mathcal{I})_{\hat{\gamma}}, \hat{V} \otimes \hat{W} \rangle = 0 \forall \hat{W} \in A(\hat{\gamma})\}$ .



at the end-points  $X : [t_0, t_1] \rightarrow V(\gamma)$  as follows:

$$X(t) := \begin{cases} J(t) & t_0 \leq t \leq \tau \\ 0 & \tau \leq t \leq t_1 \end{cases}$$

Then, preserving the notation  $\hat{X}$  for the lift of an infinitesimal deformation  $X$  and denoting by  $(\hat{J}, \lambda)$  a Jacobi pair belonging to  $J$ , in view of equation (3.12) we have

$$\begin{aligned} \left\langle (d^2\mathcal{I})_{\hat{\gamma}}, \hat{X} \otimes \hat{X} \right\rangle &= \int_{t_0}^{\tau} \left\langle (d^2\mathcal{L}')_{\hat{\gamma}}, \hat{J} \otimes \hat{J} \right\rangle dt = \\ &= \left[ \bar{\pi}_i \hat{J}^i \right]_{t_0}^{\tau} = 0 \end{aligned}$$

At the same time, if  $W$  is any infinitesimal deformation of  $\gamma$  vanishing at the end-points and such that  $W(\tau) \neq 0$ , we have also

$$\begin{aligned} \left\langle (d^2\mathcal{I})_{\hat{\gamma}}, \hat{X} \otimes \hat{W} \right\rangle &= \int_{t_0}^{\tau} \left\langle (d^2\mathcal{L}')_{\hat{\gamma}}, \hat{J} \otimes \hat{W} \right\rangle dt = \\ &= \bar{\pi}_i(\tau) \hat{W}^i(\tau) \end{aligned}$$

Since, by hypothesis,  $J(t) \neq 0$  for every  $t \in (t_0, \tau)$ , the uniqueness of the solution of the “time-reversed” Cauchy problem (3.9) in  $\gamma(\tau)$  implies that  $\bar{\pi}_i(\tau) \neq 0$  for at least one value of the index  $i$ . Therefore

$$\left\langle (d^2\mathcal{I})_{\hat{\gamma}}, \hat{X} \otimes \hat{W} \right\rangle \neq 0$$

showing that  $\hat{X}$  does not belong to the kernel of  $(d^2\mathcal{I})_{\hat{\gamma}}$ .

This completes the proof, as the indefinite character of the quadratic form  $(d^2\mathcal{I})_{\hat{\gamma}}$  is made apparent simply by looking at its action on any linear combination of the form  $\alpha \hat{X} + \hat{W}$ , for any  $\alpha \in \mathbb{R}$ .  $\square$

As an immediate consequence of Theorem 3.1 we have the following

**Proposition 3.2 (Necessary conditions).** *Suppose the extremal closed arc  $\gamma : [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  is a (local) minimum of the functional  $\mathcal{I}[\gamma] = \int_{\hat{\gamma}} \mathcal{L} dt$ . Then, for every  $\tau \in (t_0, t_1)$ , there cannot be any point  $\gamma(\tau)$  conjugate to  $\gamma(t_0)$ .*

#### 4. NECESSARY AND SUFFICIENT CONDITIONS

**4.1. Riccati and Jacobi equations.** So far we have separately proved a sufficient and a necessary condition for a given regular normal extremal  $\gamma$  to be a minimum; we shall now merge them together into a necessary and sufficient one. In the event, we will prove that, whenever no conjugate point is present, the solutions of Jacobi equation can be used to build a global solution of Riccati equation, thereby allowing to apply Theorem 2.3 to the whole interval  $[t_0, t_1]$ .

However, in order to do so, we shall need to strengthen the hypothesis of normality of  $\gamma$  by requiring the latter to be *locally normal*.

First of all, recalling that the closed arc  $\gamma: [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  is the restriction to the closed interval  $[t_0, t_1]$  of an admissible section defined on some open neighborhood  $(a, b) \supset [t_0, t_1]$ , we consider the following technical argument

**Lemma 4.1.** *Let  $\gamma: [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  be a locally normal extremal and suppose the matrix  $G_{AB}$  is non-singular at each  $t \in [t_0, t_1]$ . If along  $\gamma$  there is no point conjugate to  $\gamma(t_0)$ , then there exists a  $t^* > t_1$  such that the absence of conjugate points may be extended over a wider interval  $[t_0, t^*]$ .*

*Proof.* Consider the family of Jacobi pairs  $(\hat{X}_{(k)}, \lambda_{(k)})$ ,  $k = 1, \dots, n$ , obtained as solutions of equations (3.9) with initial data

$$X_{(k)}^i(t_0) = 0 \quad , \quad \bar{\pi}_{i(k)}(t_0) = \delta_{ik}$$

The non-existence of conjugate points along  $\gamma$  is easily seen to be equivalent to the condition  $\det(X_{(k)}^i(t)) \neq 0$  for all  $t \in (t_0, t_1]$ .

If that is not the case, there would be some  $\tau \in (t_0, t_1]$  at which the homogenous system  $a^k X_{(k)}^i(\tau) = 0$  would admit a non-null solution  $a^1, \dots, a^n$ . The fields  $\hat{X} := a^k \hat{X}_{(k)}$ ,  $\lambda = a^k \lambda_{(k)}$  would then constitute a Jacobi pair satisfying the conditions  $\lambda(t_0) \neq 0$ ,  $\hat{X}(t_0) = \hat{X}(\tau) = 0$ . On the other hand,  $\hat{X}$  cannot be identically zero over the whole interval  $[t_0, \tau]$ : if it were so, the 1-form  $\lambda$  would satisfy the equations

$$\begin{aligned} \left( \frac{DX^i}{Dt} \right)_\gamma &= M^{ij} \bar{\pi}_j = 0 \quad \implies \quad \bar{\pi}_j \left( \frac{\partial \psi^j}{\partial z^B} \right)_{\hat{\gamma}} = 0 \\ \left( \frac{D\bar{\pi}_i}{Dt} \right)_\gamma &= 0 \end{aligned} \quad \forall t_0 \leq t \leq \tau$$

contradicting the local normality of  $\gamma$ .

To sum up,  $\hat{X}$  would be a non-zero Jacobi vector field vanishing at both  $t_0$  and  $\tau$ , which clashes with the assumption of non-existence of conjugate points along  $\gamma$ .

By continuity, this implies the condition  $\det(X_{(k)}^i(t)) \neq 0$  for all  $t \in (t_0, t^*]$  with  $t^* \in (t_1, b)$  sufficiently close to  $t_1$ . The absence of conjugate points holds therefore in a wider interval  $[t_0, t^*]$ .  $\square$

Coming back to the formulation of the necessary and sufficient conditions for minimality, we are now able to state the following

**Proposition 4.1.** *Let  $\gamma: [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  be a locally normal extremal and suppose the matrix  $G_{AB}$  is non-singular at each  $t \in [t_0, t_1]$ . If no pair of conjugate points exists on  $\gamma$ , the Riccati equation (2.28) admits a symmetric solution throughout the interval  $[t_0, t_1]$ .*

*Proof.* As usual, we regard  $\gamma$  as the restriction of an admissible section defined on an open interval  $(a, b) \supset [t_0, t_1]$ . Let  $t^* \in (t_1, b)$  and consider a family of solutions  $(\hat{X}_{(k)}, \lambda_{(k)})$  of equations (3.9), obtained imposing the initial conditions  $X_{(k)}^i(t^*) = 0$  and  $(\bar{\pi})_{i(k)}(t^*) = \delta_{ik}$ .

In view of Lemma 4.1, whenever  $t^*$  is chosen sufficiently close to  $t_1$ , the absence of conjugate points implies the requirement  $\det(X_{(k)}^i(t)) \neq 0$  for all  $t \in [t_0, t^*]$ .

A comparison between the Hamiltonian system (3.9) and the linearization (2.29) of Riccati equation shows that we can now assume the identifications

$$(4.1) \quad K^i_j(t) := X^i_{(j)}(t), \quad E_{ij}(t) := \bar{\pi}_{i(j)}(t)$$

As a consequence, the matrix  $K^i_j(t)$  is non-singular everywhere on  $[t_0, t^*)$  and therefore, as we have seen in § 2.3, the tensor  $C_{ij} = E_{ir} (K^{-1})^r_j$  represents a solution of Riccati equation (2.28) all over the interval  $[t_0, t^*) \supset [t_0, t_1]$ .

In order to complete the proof, we now only need to show that the stated  $C_{ij}$  is also a symmetric tensor. To this end, we first observe that the matrix  $R^{ip} := K^i_j (E^{-1})^{jp}$  is perfectly meaningful in a neighborhood  $(t^* - \delta, t^*]$  and satisfies the relations

$$R^{ip}(t^*) = 0, \quad R^{ip} C_{pq} = K^i_j (E^{-1})^{jp} E_{pr} (K^{-1})^r_q = \delta^i_p \quad \forall t < t^*$$

The matrix  $R^{ip}$  is therefore symmetric at  $t = t^*$ . Moreover, on account of equations (3.9), it satisfies the equation

$$\frac{DR^{ip}}{Dt} = \frac{DK^i_j}{Dt} (E^{-1})^{jp} + K^i_j \frac{D(E^{-1})^{jp}}{Dt} = M^{ir} - R^{il} N_{lk} R^{kp}$$

which is again of the Riccati-type (2.28), with the roles of the matrices  $M^{ij}, N_{ij}$  interchanged. Exactly as in Theorem 2.2, this establishes the symmetry of  $R^{ip}$  in a neighborhood of  $t = t^*$ .

For each  $t \in (t^* - \delta, t^*)$  the matrix  $C_{ij}(t) = (R^{ij}(t))^{-1}$  is therefore symmetric. Once again, on account of the linearity of equation (2.25b), we conclude that  $C_{ij}(t)$  is symmetric over the whole interval  $[t_0, t^*) \supset [t_0, t_1]$ .  $\square$

Collecting all the above arguments, we are now able to state the following

**Theorem 4.1 (Necessary and sufficient conditions).** *Suppose the closed arc  $\gamma: [t_0, t_1] \rightarrow \mathcal{V}_{n+1}$  is a locally normal extremal of the functional  $\mathcal{I}[\gamma] = \int_{\tilde{\gamma}} \mathcal{L} dt$  with respect to the class of deformations vanishing at the end-points and let  $\tilde{\gamma}$  be its (unique) lift to  $\mathcal{C}(\mathcal{A})$  solving Pontryagin equations (3.2). Denote by  $\mathcal{H}(t, q^i, z^A, p_i) = p_i \psi^i(t, q^i, z^A) - \mathcal{L}(t, q^i, z^A)$  the Pontryagin Hamiltonian associated with the given Lagrangian and let*

$$G_{AB}(t) := - \left( \frac{\partial^2 \mathcal{H}}{\partial z^A \partial z^B} \right)_{\tilde{\gamma}}$$

*The section  $\gamma$  is a minimum of the functional  $\mathcal{I}[\gamma]$  if and only if, for any  $t \in [t_0, t_1]$ , the matrix  $G_{AB}$  is positive definite and there is no point conjugate to  $\gamma(t_0)$ .*

The proof should, at this time, be quite straightforward and is left to the reader.

## REFERENCES

- [1] E. Massa, D. Bruno and E. Pagani On the calculus of variations with constraints I: the first variation, *arXiv*: 0705.2362v1
- [2] E. Massa and E. Pagani, A new look at Classical Mechanics of constrained systems, *Ann. Inst. Henri Poincaré, Physique théorique*, Vol. **66**, 1997, pp. 1–36.
- [3] E. Massa, S. Vignolo and D. Bruno, Non-holonomic Lagrangian and Hamiltonian Mechanics: an intrinsic approach, *J. Phys. A: Math. Gen.* **35**, 6713–6742 (2002).
- [4] M. Crampin, Tangent Bundle Geometry for Lagrangian Dynamics, *J. Phys. A: Math. Gen.*, **16**, 3755–3772 (1983)

- [5] W. Sarlet, F. Cantrijn and M. Crampin, A New Look at Second Order Equations and Lagrangian Mechanics, *J. Phys. A: Math. Gen.*, **17**, 1999–2009 (1984).
- [6] G.A. Bliss, *Lectures on the calculus of the variations*, The University of Chicago Press, Chicago (1946).
- [7] C. Lanczos, *The variational principles of mechanics*, University of Toronto Press, Toronto (1949) (Reprinted by Dover Publ. (1970)).
- [8] W. Hurewicz, *Lectures on ordinary differential equations*, John Wiley & Sons, Inc., New York and Cambridge, Mass. (1958). University of Toronto Press, Toronto (1949) (Reprinted by Dover Publ. (1970)).
- [9] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze and E.F. Mishchenko, *The mathematical theory of optimal process*, Interscience, New York (1962).
- [10] I.M. Gelfand and S.V. Fomin, *Calculus of variations*, Prentice-Hall Inc., Englewood Cliffs (1963).
- [11] S. Sternberg, *Lectures on Differential Geometry*, Prentice Hall, Englewood Cliffs, New Jersey (1964).
- [12] H. Rund, *The Hamilton-Jacobi theory in the calculus of variations*, Van Nostrand, London (1966).
- [13] M.R. Hestenes, *Calculus of variations and optimal control theory*, Wiley, New York London Sydney (1966).
- [14] J.F. Pommaret, *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon & Breach, New York (1978).
- [15] L. C. Young *Lectures on the Calculus of Variations and Optimal Control Theory* (second edition), AMS Chelsea Publishing, New York (1980).
- [16] P. Griffiths, *Exterior differential systems and the calculus of variations*, Birkhauser, Boston (1983).
- [17] F. W. Warner, *Foundations of Differential Manifolds and Lie Groups*, Springer-Verlag, New York (1983).
- [18] S. Benenti, *Relazioni simplettiche*, Pitagora Editrice, Bologna (1988).
- [19] M. de Leon and P.R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics*, North Holland, Amsterdam (1989).
- [20] D.J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society, Lecture Note Series 142, Cambridge University Press (1989).
- [21] M. Giaquinta and S. Hildebrandt, *Calculus of variations I, II*, Springer-Verlag, Berlin Heidelberg New York (1996).
- [22] A. A. Agrachev and Yu.L. Sachov, *Control Theory from the Geometric Viewpoint*, Springer-Verlag, Berlin Heidelberg New York (2004).
- [23] V. I. Arnold, *Dynamical Systems III, Encyclopaedia of Mathematical Sciences*, Springer-Verlag, Berlin Heidelberg New York (1985).
- [24] R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, AMS, Math. Surveys and Monographs, Vol. 91 (2000).
- [25] H. Sagan, *Introduction to the calculus of variations*, McGraw-Hill Book Company, New York (1969)

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